New approach to stochastic modeling of the problems in atmospheric optics

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Received October 5, 2005

The problems of statistical modeling of the optical radiation transfer through stochastic scattering and absorbing media are considered. It is assumed that the spatial variations of optical parameters of these media are random. The weighting algorithm proposed in the paper enables one to optimize modeling. In this approach, random trajectories are constructed for a determined medium, while random variations of its optical parameters are considered by calculating special weighting factors. In a great number of practically important problems in optics of stochastic media, the algorithm allows a simple numerical or sometimes analytical integration of the corresponding random estimates, which are necessary in calculating average characteristics of the radiation field. The algorithm developed allows one to construct numerical models of the electromagnetic radiation in randomly inhomogeneous media mostly intended for solving problems of solar radiation transfer through the stochastic continuous cloudiness.

Introduction

As known, the transfer of optical radiation through the absorbing and scattering media can be described in the approximation of the geometrical optics by the following integral equation (Ref. 1)

$$f(\mathbf{x}) = \int_X k(\mathbf{x}',\mathbf{x}) f(\mathbf{x}')\mathrm{d}\mathbf{x}' + \psi(\mathbf{x}), \qquad (1)$$

where $f(\mathbf{x})$ is the collision density; $\mathbf{x} = (\mathbf{r}, \boldsymbol{\omega})$ and $\mathbf{x}' = (\mathbf{r}', \boldsymbol{\omega}')$ are the points of the phase space $X = \{\mathbf{r} \in R \subset R^3, \boldsymbol{\omega} = (a,b,c) \in \Omega = (a^2 + b^2 + c^2 = 1)\}$; $\psi(\mathbf{x})$ is the source distribution density; $\int_X \psi(\mathbf{x})\mathrm{d}\mathbf{x} = 1$. The exact form of the kernel $k(\mathbf{x}',\mathbf{x})$ in Eq. (1) is determined by the problem type and its boundary conditions. The task is to estimate linear functionals of the form

$$I_\varphi = (f,\varphi) = \int_X f(\mathbf{x})\varphi(\mathbf{x})\mathrm{d}\mathbf{x}, \qquad (2)$$

where $\varphi(\mathbf{x}) \geq 0$ is the so-called instrumental function that determines the form of the sought characteristic of the optical field. Calculation of the I_φ functionals by the Monte Carlo method is related to modeling the homogeneous Markovian chains, whose states are presented by the sequence $\mathbf{x}_0, \mathbf{x}_1, \ldots, \mathbf{x}_n$, where the initial state is determined by the density $r_0(\mathbf{x})$, while the transition from the state $\mathbf{x}_{i-1}$ to $\mathbf{x}_i$ is determined by the density of the transition $r(\mathbf{x}_{i-1}, \mathbf{x}_i)$. If

$$r_0(\mathbf{x}) = \psi(\mathbf{x}) \text{ and } r(\mathbf{x}_{i-1},\mathbf{x}_i) = k(\mathbf{x}_{i-1},\mathbf{x}_i),$$

then, the chain $\{\mathbf{x}_n\}$ is the physical chain of collisions and the corresponding modeling method is called the analog or physical one. In this case,

$$I_\varphi = M\xi, \quad \xi = \sum_{n=0}^{N} \varphi(\mathbf{x}_n),$$

where N is the random number of the chain truncation; M is the symbol of mathematical expectation. In the case of non-analog modeling, one calculates the random weight

$$Q_n = Q_{n-1}\frac{k(\mathbf{x}_{n-1},\mathbf{x}_n)}{r(\mathbf{x}_{n-1},\mathbf{x}_n)}, \quad Q_0 = \frac{\psi(\mathbf{x}_0)}{r_0(\mathbf{x}_0)} \qquad (3)$$

with the requirements that $r_0(\mathbf{x}) \neq 0$ at $\psi(\mathbf{x}) \neq 0$ and $r(\mathbf{x}',\mathbf{x}) \neq 0$ at $k(\mathbf{x}',\mathbf{x}) \neq 0$, and for the functional I_φ one calculates the unbiased random estimate

$$\xi = \sum_{n=0}^{N} Q_n \varphi(\mathbf{x}_n), \qquad (4)$$

so that $M\xi = I_\varphi$. The general theory of constructing the effective weighted estimates of the type (4) can be found in Ref. 2.

Now consider a stochastic problem, where one or several kernel-affecting parameters $\sigma = (\sigma_1, \sigma_2, \ldots, \sigma_s)$ and the collision density $f(\mathbf{x})$ are random functions of space (random fields). The problem can be solved by calculating the random quantities $\xi(\omega, \sigma)$ set along the paths ω of the random process modeled such that

$$M\xi[(\omega,\sigma)|\sigma] = I_\varphi(\sigma).$$

The paths ω depend on σ. The sought functional is determined as

0235-6880/05/12 951-04 $02.00

$$I = \langle I_\varphi(\sigma) \rangle ,$$

where $\langle\rangle$ denote statistical averaging over the distribution of the random field σ. One of the most important stochastic problems of atmospheric optics is simulation of solar radiation transfer through a cloudy atmosphere. This problem is important for two reasons. First, clouds that always cover the major part of the Earth's surface are one of the main factors determining the radiative transfer in the "atmosphere — underlying surface" system and the inflow of solar energy to the Earth's surface. Second, any cloudiness has a stochastic structure, which can tolerably be described only by statistical modeling. This paper addresses some questions of applying statistical modeling to solution of this stochastic problem.

Statement of the problem

Consider the process of particle (photon) transfer in the phase space $X = R \times \Omega$, $\mathbf{x}(\mathbf{r}, \omega) \in X$ of the coordinates $\mathbf{r} \in R = (-\infty, \infty) \times [h, H]$ and directions $\boldsymbol{\omega}(\mu,\varphi) \in \Omega = [-1,1] \times [0,2\pi]$. The unit vector $\boldsymbol{\omega}$ ($|\boldsymbol{\omega}| = 1$) along the particle travel direction is determined by the characteristics $\mu = \cos\theta$ and φ. Here, θ is the angle between the vector $\boldsymbol{\omega}$ and the axis OZ, φ is the azimuth angle, i.e., the angle between the axis OX and the vector $\boldsymbol{\omega}_\perp$ (projection of $\boldsymbol{\omega}$ onto the plane $z = 0$); and Ω_- and Ω_+ are the spaces of the unit vectors $\boldsymbol{\omega}$, for which $\mu \in [-1, 0]$ and $\mu \in [0, 1]$, respectively.

The effect of the scattering and absorbing medium on the transfer of photons is determined by macroscopic cross sections of scattering $\Sigma_s(\mathbf{r})$, absorption $\Sigma_a(\mathbf{r})$, and by the scattering phase function $g(\mathbf{r}, \tilde{\mu})$, where $\tilde{\mu} = (\boldsymbol{\omega}', \boldsymbol{\omega})$ is the cosine of the angle between the directions $\boldsymbol{\omega}'$ and $\boldsymbol{\omega}$ of the particle trajectory prior to and after the scattering event, respectively. The function $g(\mathbf{r}, \tilde{\mu})$ has the following property:

$$\int_{-1}^{+1} g(\mathbf{r}, \tilde{\mu}) \mathrm{d}\tilde{\mu} = 1.$$

Hereinafter, we shall use the following characteristics:

$$\Sigma(\mathbf{r}) = \Sigma_a(\mathbf{r}) + \Sigma_s(\mathbf{r}),$$

which is the macroscopic cross section of extinction; the single scattering albedo (the probability of a particle to survival after a collision),

$$q(\mathbf{r}) = \Sigma_s(\mathbf{r}) / \Sigma(\mathbf{r}) ,$$

and the optical path length between the points $\mathbf{r}'$ and $\mathbf{r}$,

$$\tau(\mathbf{r}', \mathbf{r}) = \int_0^l \Sigma(\mathbf{r}' + \boldsymbol{\omega} s) \mathrm{d}s ,$$

where

$$l = |\mathbf{r} - \mathbf{r}'| , \quad \boldsymbol{\omega} = (\mathbf{r} - \mathbf{r}') / |\mathbf{r} - \mathbf{r}'| .$$

In our case, the source of particles is a unidirectional infinitely wide flux of photons incident on the upper boundary of the scattering layer at $z = H$. Such a source is described by the function $S(\mathbf{r}, \boldsymbol{\omega}) = \pi S_0 \delta(z - H) \delta(\boldsymbol{\omega} - \boldsymbol{\omega}_0)$, where πS_0 is the solar constant at the chosen wavelength; $\boldsymbol{\omega}_0$ is the unit vector along the direction of solar radiation incidence onto the layer boundary.

Under such conditions, we have in Eq. (1) that

$$\psi(\mathbf{x}) = \psi(\mathbf{r}, \boldsymbol{\omega}) = \pi S_0 \delta(z - H) \delta(\boldsymbol{\omega} - \boldsymbol{\omega}_0),$$

and the kernel of Eq. (1) takes the form (Ref. 1):

$$k(\mathbf{r}', \boldsymbol{\omega}' \to \mathbf{r}, \boldsymbol{\omega}) = \frac{q(\mathbf{r}') g\left(\mathbf{r}', (\boldsymbol{\omega}', \boldsymbol{\omega})\right)}{2\pi} \times \frac{1}{|\mathbf{r} - \mathbf{r}'|^2} \Sigma(\mathbf{r}) e^{-\tau(\mathbf{r}', \mathbf{r})} \delta\left(\boldsymbol{\omega} - \frac{\mathbf{r} - \mathbf{r}'}{|\mathbf{r} - \mathbf{r}'|}\right). \quad (5)$$

As we proceed to stochastic problem, we assume that one or several (in any combination) of the initial optical parameters in Eq. (5), namely, $q(\mathbf{r})$, $\Sigma(\mathbf{r})$, and $g(\mathbf{r}, \tilde{\mu})$ are random functions of space. In the previous section, these functions are designated in general as σ.

Thus, hereinafter, we limit our consideration to statistical modeling of the radiation transfer through an isolated stochastic cloudiness ignoring scattering and absorption in the atmosphere below and above the clouds. Note that the radiation model of isolated cloudiness can easily be incorporated into the radiation model of the aerosol atmosphere, which is based on the already used in practice computer-based automated system for statistical modeling of the problems in atmospheric optics (Ref. 3).

Weighting algorithm of stochastic modeling

The standard approach to solution of the above stochastic problem is based on the method of double randomization (see, for example, Ref. 1), which follows from the relation

$$\langle I_\varphi(\sigma) \rangle = \langle M_\omega [\xi(\omega, \sigma) | \sigma] \rangle = M_{(\omega\sigma)} \xi(\omega, \sigma),$$

where ω stands for the random path of the Markovian chain. In this case, the algorithm for calculating I involves:

1. Simulation of realization of the random field σ;
2. For each realization of σ, simulation of $n (n \geq 1)$ conditionally independent paths of the Markovian chain ω;
3. Calculation of the corresponding random estimates $\xi(\omega, \sigma)$.

With such an approach, there arise principle computational difficulties resulting in a significantly

longer time of computations as compared with that in the case of a deterministic problem. The main difficulty is simulation of the photon paths in a randomly inhomogeneous 3D medium. In other words, every time in modeling the successive collision point, one has to solve a laborious problem of numerical analysis, namely, solution of the equation with respect to the parameter l

$$\int_0^l \Sigma(\mathbf{r}' + \boldsymbol{\omega} s)\mathrm{d}s = -\ln\alpha, \qquad (6)$$

where α is the random number uniformly distributed over the interval (0, 1). To solve this equation, we usually use a discrete approximation of the random function $\Sigma(\mathbf{r}' + \boldsymbol{\omega}\sigma)$, $s \geq 0$. In the general case, this approach can result in uncontrolled errors in solution of Eq. (6). Reduction of the approximation step h_s of the function $\Sigma(\mathbf{r}' + \boldsymbol{\omega}\sigma)$ along the direction $\boldsymbol{\omega}$ aimed at improving the approximation accuracy inevitably leads to the growth of the number of arithmetic operations needed to evaluate the root of Eq. (6) at the rate proportional to $1/h_s$. All this makes the above approach to simulation of the photon paths in randomly inhomogeneous media difficult to use for the entire series of practically important stochastic problems. For instance, this is true in the case of using spectral models of a random field Σ, in which the function $\Sigma(\mathbf{r}' + \boldsymbol{\omega}\sigma)$ is a continuous random process along the arbitrary direction $\boldsymbol{\omega}$.

The weighting algorithm considered below allows us to eliminate the laborious solution of Eq. (6), which we would need to perform for each of the collision events, and to reduce modeling to the case analogous to the deterministic problem. The idea of this algorithm consists in the following. The estimates of the type (4) of the unknown functional I can be found from the same random photon paths for different realizations of the random field $\Sigma(\mathbf{r})$ with the account of random weights (3) that remove the arising bias. Namely, the paths built for a certain determined function $\Sigma_0(\mathbf{r})$ can be used for averaging of the functional $I_\varphi(\Sigma)$ over the realizations of $\Sigma(\mathbf{r})$, if after each transition $\mathbf{x}' \to \mathbf{x}$, the corresponding photon weight will be multiplied by the quantity $k(\mathbf{x}',\mathbf{x};\Sigma)/k(\mathbf{x}',\mathbf{x};\Sigma_0)$.

Assume that

$$\omega_n = \{(\mathbf{x}_0,\mathbf{x}_1,\ldots,\mathbf{x}_n);\ \mathbf{x}_i = (\mathbf{r}_i,\boldsymbol{\omega}_i);\ i = \overline{0,n}\}$$

is a random n-segment path built for the transition density $r(\mathbf{x}',\mathbf{x}) = k(\mathbf{x}',\mathbf{x};\Sigma_0)$. Now, the weighting coefficient $Q_n(\Sigma)$ that corresponds to the realization of $\Sigma(\mathbf{r})$ is calculated by the formula

$$Q_n(\Sigma) = \frac{\prod_{i=1}^{n-1}\Sigma_s(\mathbf{r}_i)}{\prod_{i=1}^{n}\Sigma_{s,0}(\mathbf{r}_i)}\frac{\Sigma(\mathbf{r}_n)}{\Sigma_0(\mathbf{r}_n)}e^{-\sum_{i=1}^{n}(\tau(\mathbf{r}_{i-1},\mathbf{r}_i)-\tau_0(\mathbf{r}_{i-1},\mathbf{r}_i))}, \qquad (7)$$

where

$$\tau_0(\mathbf{r}_{i-1},\mathbf{r}_i) = \int_0^{|\mathbf{r}_{i-1}-\mathbf{r}_i|}\Sigma_0(\mathbf{r}_{i-1}+s\boldsymbol{\omega}_{i-1})\mathrm{d}s;$$

and (8)

$$\tau(\mathbf{r}_{i-1},\mathbf{r}_i) = \int_0^{|\mathbf{r}_{i-1}-\mathbf{r}_i|}\Sigma(\mathbf{r}_{i-1}+s\boldsymbol{\omega}_{i-1})\mathrm{d}s.$$

From Eqs. (7) and (8) it follows that to calculate $\xi(\Sigma)$ along the path ω_n built for the density $\mathbf{k}(\mathbf{x}', \mathbf{x}; \Sigma_0)$, we need to obtain realizations of the random function $\Sigma(\mathbf{r})$ only at the points $\mathbf{r}_1, \ldots, \mathbf{r}_n$, i.e., $\Sigma(\mathbf{r}_1), \ldots, \Sigma(\mathbf{r}_n)$, and determine the values of integrals (8) of the functions $\Sigma(\mathbf{r}_i + \boldsymbol{\omega}_i s)$, $i = 1, \ldots, n$, along $\boldsymbol{\omega}_i = (\mathbf{r}_i - \mathbf{r}_{i-1})/|\mathbf{r}_i - \mathbf{r}_{i-1}|$ in the segments $(\mathbf{r}_{i-1},\mathbf{r}_i)$.

Thus, the problem of averaging the estimate of $\xi(\Sigma)$ over realizations of the continuous random field Σ is reduced to the averaging over realizations of the random vectors $\{\Sigma(\mathbf{r}_i)\}_{i=\overline{1,n}}$ and $\{\tau(\mathbf{r}_{i-1},\mathbf{r}_i;\Sigma)\}_{i=\overline{1,n}}$. In many cases it considerably reduces computation time. The computation time gain depends on the model of the random field $\Sigma(\mathbf{r})$.

As an example, consider a popular stochastic model of continuous stratified cloud layer $h \leq z \leq H$ in the form of a vertically homogeneous stationary random process $\Sigma(z)$. As an approximate process $\Sigma(z)$, we can use one of the numerous spectral models (see, e.g., Ref. 4)

$$\Sigma(z) \approx \Sigma^{(k)}(z) = \overline{\Sigma(z)} + \sigma_\Sigma\sum_{j=1}^{k} a_j\sqrt{-2\ln\alpha_j}\cos(\lambda_j z + 2\pi\beta_j),$$

where α_j and β_j are the independent random quantities uniformly distributed over the interval (0, 1); $a_j^2 = 1/k$; σ_Σ^2 is variance of the marginal Σ distribution; $\lambda_j \in [0, \infty)$ are distributed with the probability density

$$S(\lambda) = \frac{2}{\pi}\int_0^\infty \cos(\lambda z)K_\Sigma(z)\mathrm{d}z,$$

$K_\Sigma(z)$ is the correlation function. In this case, the quantity $\tau(\mathbf{r}_{i-1},\mathbf{r}_i)_{i=\overline{1,n}}$ in Eq. (8) is calculated by the formula

$$\tau(\mathbf{r}_{i-1},\mathbf{r}_i) = \frac{1}{|(\boldsymbol{\omega}_{i-1},\mathbf{k})|}\Big[\overline{\Sigma}|z_{i-1}-z_i| +$$

$$+\sigma_\Sigma\sum_{j=1}^{k}\frac{a_j}{\lambda_j}\sqrt{-2\ln\alpha_j}\left[\sin(\lambda_j z_{i-1}+2\pi\beta_j)-\sin(\lambda_j z_i+2\pi\beta_j)\right]\Big].$$

Here, $\mathbf{k} = (0, 0, 1)$.

As the marginal distribution of Σ, a truncated normal distribution is often used with the mathematical expectation $\overline{\Sigma}$ and variance σ_Σ^2, i.e., $\varphi(\Sigma) = \Phi(\Sigma) = 0$ at $\Sigma < \Sigma_{\min}$ and

$$\varphi(\Sigma) = \frac{1}{\sigma_{\Sigma,u}(1-\tau_a)}\varphi_u\left(\frac{\Sigma-\bar{\Sigma}_u}{\sigma_{\Sigma,u}}\right),$$

$$\Phi(\Sigma) = \frac{\Phi_u\left(\frac{\Sigma-\bar{\Sigma}_u}{\sigma_{\Sigma,u}}\right)-\tau_a}{1-\tau_a},$$

where $\varphi_u(x)$ and $\Phi_u(x)$ are, respectively, the density and the distribution function of the normal distribution with the parameters $\bar{\Sigma}_u$, $\sigma_{\Sigma,u}$, and $\tau_a = \Phi_u[(\Sigma_{\min}-\bar{\Sigma}_u)/\sigma_{\Sigma,u}]$. To find the values $\bar{\Sigma}_u$ and $\sigma_{\Sigma,u}$, one can use the equations

$$\alpha_1 = \bar{\Sigma}_u + \sigma^2_{\Sigma,u}\varphi(\Sigma_{\min}),$$

$$\alpha_2 = \bar{\Sigma}^2_u + \sigma^2_{\Sigma,u}\varphi(\Sigma_{\min})(\Sigma_{\min}+\bar{\Sigma}_u) + \sigma^2_{\Sigma,u}. \qquad (9)$$

Using Eqs. (9), the values of $\bar{\Sigma}$, σ_Σ, and Pearson's tables expressing the $\bar{\Sigma}$ and σ_Σ values through α_1 and α_2, it is easy to find $\bar{\Sigma}_u$ and $\sigma_{\Sigma,u}$.

Concluding remarks

This weighting algorithm allows us to eliminate the laborious procedure of modeling the random photon paths in randomly inhomogeneous media and thus to reduce the computation time. The suggested algorithm is easy to use. The gain in computation time depends on the random field model. To investigate this dependence, we have conducted a series of numerical experiments.

Acknowledgments

The study has been supported by the Russian Foundation for Basic Research (No. 03-01-0040) and The Leading Scientific Schools (NSh-1271, 2003).

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