SPECTRAL AND DYNAMIC CHARACTERISTICS OF THE ADAPTIVE IMAGING SYSTEMS

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Two aspects of the adaptive system operation, i.e., the effect of the time delay and the spectral dependence of the quality of phase correction are analyzed. Numerical analysis is based on the most reliable model profiles of the spectral density of fluctuations of the atmospheric refractive index.

One of the most widespread imaging systems is a ground–based astronomical telescope. When using such a telescope in the regime of adaptive compensation for atmospheric phase distortions of the wavefront, it additionally comprises such elements as the system of monitoring the phase distortions (the wavefront sensor), the computer system, and the controllable mirror incorporated in the optical system of the telescope.[1] These elements compensate for the atmospheric distortions in real time.

As any other system, the adaptive telescope has a finite frequency band in which the distortions can be effectively compensated. That is so because of the finite response time of the wavefront sensor, of the computer, and of the inertial controllable mirror. Thus the problem of determining the necessary frequency band or the maximum time delay between the measurement of the phase distortions and their subsequent compensation with the controllable mirror is quite important.

The second problem we are going to touch upon in this paper, associated with the interferometric techniques used in the wavefront sensor of the telescope. For this reason the mesurements of the phase distortions of the wavefront and, hence, the control of the profile of the adaptive mirror are made in quite a narrow spectral band around a certain radiation wavelength (cut out with the help of the interference filter). At the same time, the telescope forms the image with quite a broad wavelength band typical of the observed sources of light. The problem of effectiveness of compensation for broad–band distortions with the use of the narrow–band measurements is formulated in special–purpose publications (see Refs. 2 and 3) as the problem of "spectral characteristics of adaptive systems" or "two–colour adaptive systems".

Both these aspects have been already touched upon elsewhere by one of the authors.[2,4–7] However, as a rule, an asymptotic analysis was performed in these publications. At the same time, a number of papers are now available, which in a definite sense contradict with the results obtained earlier.[3,8,9] For this reason, we present the results of numerical calculations based on the most commonly used models of spectral density of fluctuations of the atmospheric refractive index along the vertical path.

DYNAMIC CHARACTERISTICS OF ADAPTIVE OPTICAL SYSTEMS

For our starting relation we make use of the representation of the field $V(x, \rho)$ formed in vacuum in the arbitrary plane x

$$U(x, \rho) = \int\int \mathrm{d}^2\rho' U_0(\rho') \frac{\kappa}{2\pi i(x - x_0)} \exp\left(i\kappa \frac{(\rho - \rho')^2}{2(x - x_0)}\right), \quad (1)$$

where $U_0(\rho) = A(\rho)\exp(- i\kappa\rho^2/2F)$ describes the initial wave incident upon the circular lens (F is its focal length). In the absence of fluctuations in the initial wave U_0 the field in the focal plane of this lens on its axis can be written as

$$I_{\mathrm{vac}}(0) = \frac{1}{\lambda^2F^2} \int\int \mathrm{d}^4\rho_{1,2} A(\rho_1) A^*(\rho_2) = \frac{(\pi R^2)^2}{\lambda^2F^2} = \frac{\Omega_F^2}{4} . \quad (2)$$

Here R is the radius of the input lens and $\Omega_F = \kappa R^2/F$.

In the case of propagation through the atmospheric turbulence[10] we have

$$\langle I(0)\rangle = \lambda^{-2}F^{-2}\int\int \mathrm{d}^4\rho_{1,2} A(\rho_1) A^*(\rho_2) \times$$

$$\times \exp\left(-\frac{1}{2}D_s(|\rho_1-\rho_2|)\right) = \frac{\pi}{\lambda^2F^2}\int\limits_0^D \rho\mathrm{d}\rho\, K(\rho) \exp\left(-\frac{1}{2}D_s\right), \quad (3)$$

where

$$K(\rho) = \begin{cases} \frac{1}{2}\left[D^2\arccos(r/D) - r\sqrt{D^2 - r^2}\right], & \text{for } r \le D \\ 0\ , & \text{for } r > D, \end{cases} \quad (4)$$

$D_s(\rho) = 6.88(\rho/r_0)^{5/3}$ is the structure function of phase and r_0 is the Fried radius. We also note here that aperture function (4) has quite a simple and efficient approximation[11]

$$\tilde{K}(x) = 1 - 1.25x + 0.25x^4\ , \ \tilde{K}(x) = \frac{2}{\pi} K(r/D)\ .$$

To estimate the effectiveness of the optical system, we use the Strehl factor

$$\mathrm{St} = \frac{\int\limits_0^1 \mathrm{d}x x\, \tilde{K}(x) \exp\{- 3.44(D/r_0)^{5/3}x^{5/3}\}}{\int \mathrm{d}x x\, \tilde{K}(x)}, \quad (5)$$

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where

$$\tilde{K}(x) = \frac{2}{\pi}\left[\arccos x - x\sqrt{1-x^2}\right],\ 0 \le x \le 1 .$$

Usually the traditional optical systems use the data of the current phase measurements to correct the distortions. Since the frequency band of any dynamic system is finite, the temporal delay between these measurements and control[2] takes place. Let us introduce the notion of time constant of the delay τ.

The traditional adaptive system with finite time delay may be classified as a system with "constant delay". To estimate the phase $\hat{S}$ at the time moment $(t + \tau)$ in such a system we use the data of preceding measurements, i.e.,

$$\hat{S}(\mathbf{r},\ t + \tau) = S(\mathbf{r},\ t) .$$

Then the phase term in the integrand, characterizing the residual distortions, can be written in the form

$$\exp\left\{-D_s(v\tau) - \left[D_s(\rho_1-\rho_2) - \frac{1}{2}D_s(\rho_1-\rho_2+v\tau) - \frac{1}{2}D_s(\rho_1-\rho_2-v\tau)\right]\right\}. \quad (6)$$

Substituting the phase term in Eq. (3) by expression (6), we obtain

$$\langle I(r)\rangle = \frac{e^{-D_s(vt)}}{\lambda^2F^2}\int\int d^2\rho\, K(\rho) \times \exp\left\{-\left[D_s(\rho) - \frac{1}{2}D_s(\rho + v\tau) - \frac{1}{2}D_s(\rho - v\tau)\right]\right\} . \quad (7)$$

Further we make use of the asimptotic behavior of integrand (6) to obtain:

$$\langle I(r)\rangle = e^{-D_s(v\tau)}\, I_{\text{vac}}(0) + \left[1 - e^{-D_s(v\tau)}\right]\langle I(0)\rangle , \quad (8)$$

where $\langle I(0)\rangle$ is given by Eq. (3), i.e., corresponds to a system without correction. Finally we have

$$\mathrm{St}(\tau) = \mathrm{St} + e^{-D_s(v\tau)}[1 - \mathrm{St}] , \quad (9)$$

where St is determined by Eq. (5). It can be easy seen from Eq. (9) that $\mathrm{St} \to 1$ as $D_s(v\tau) \to 0$ and as $D_s(v\tau) \to \infty$ as $\mathrm{St}(\tau) \to \mathrm{St}$, as in the system without correction.

It is possible to describe the phase distortions of the optical wave expanding them in a series in the orthogonal modes.[11] In addition, in calculating the average values

$$\langle[\ldots]^2\rangle = \langle\{[S(\rho_1, t+\tau) - S(\rho_1, t)] - [S(\rho_2, t+\tau) - S(\rho_2, t)]\}^2\rangle ,$$

we make use of the expansion in the Taylor series

$$S(\rho + v\tau,\ t) = S(\rho,\ t) + \nabla_\rho S(\rho,\ t)\, v\tau \quad (10)$$

to obtain

$$\langle[\ldots]^2\rangle = \tau^2\langle\{\nabla_\rho S(\rho_1,\ t)\,\mathbf{v} - \nabla_\rho S(\rho_2,\ t)\,\mathbf{v}\}^2\rangle \quad . \quad (11)$$

Using the expansion in a system of the Zernike polinominals[12]

$$S(\mathbf{r},\ t) = \sum_{j=1}^{\infty} a_j\, F_j(\mathbf{r}/R) ,\ \nabla S(\mathbf{r},\ t) = \sum_{j=2}^{\infty} a_j\, \nabla F_j(\mathbf{r}/R) , \quad (12)$$

where $a_j = \int\int d^2\rho\, A(\rho)\, S(R\,\rho,\ \theta)\, F_j(\rho,\ \theta)$, $\rho = r/R$,

we obtain

$$\langle[\ldots]^2\rangle = \tau^2\left[v_x^2\left\langle\left(\frac{\partial S(\mathbf{r}_1)}{\partial x} - \frac{\partial S(\mathbf{r}_2)}{\partial x}\right)^2\right\rangle + v_y^2\left\langle\left(\frac{\partial S(\mathbf{r}_1)}{\partial y} - \frac{\partial S(\mathbf{r}_2)}{\partial y}\right)^2\right\rangle + 2v_xv_y\left\langle\left(\frac{\partial S(\mathbf{r}_1)}{\partial x} - \frac{\partial S(\mathbf{r}_2)}{\partial x}\right)\left(\frac{\partial S(\mathbf{r}_1)}{\partial y} - \frac{\partial S(\mathbf{r}_2)}{\partial y}\right)\right\rangle\right] . \quad (13)$$

Using the conditions $v_x^2 = v_y^2 = v^2/2$ and $\langle a_4^2\rangle = \langle a_5^2\rangle = \langle a_6^2\rangle$, we find

$$\langle[\ldots]^2\rangle = 48\,\frac{v^2\tau^2}{R^4}\,\langle a_4^2\rangle\,\rho^2\left[1 + \frac{\sin 2\theta}{2}\right], \quad (14)$$

where $\langle a_4^2\rangle = 0.0232(D/r_0)^{5/3}$, $\rho = \rho(\rho,\ \theta)$. Relation (14) is obtained from Eq. (12) retaining the terms up to the sixth mode. The representation $D_s(\rho) = 6.88(\rho/r_0)^{5/3}$ is well known for the Kolmogorov model of atmospheric turbulence[10] and, strictly speaking, it uses the infinite number of modes in expansion (12). If we write the structure function of phase retaining the first six modes, we find

$$\tilde{D}_s(\rho_1,\ \rho_2) = 1.796(D/r_0)^{5/3}|\rho_1 - \rho_2|^2/R^2 + 0.42(D/r_0)^{5/3} \times [(x_1^2 - x_2^2) - (y_1^2 - y_2^2)]^2/R^4 + 0.56(D/r_0)^{5/3}(x_1\, y_2 - x_2\, y_1)^2/R^4. \quad (15)$$

The following step in developing the adaptive systems consists in constructing a "fast" adaptive system. Control in such a system is fulfilled on the basis of the algorithm

$$\hat{S}(\mathbf{r},\ t + \tau) = S(\mathbf{r},\ t) + \nabla_r S(\mathbf{r},\ t)\, v\tau . \quad (16)$$

In calculating the structure function of the residual phase distortions

$$\Delta S(\mathbf{r},\ t) = S(\mathbf{r},\ t + \tau) - S(\mathbf{r},\ t)\, v\tau ,$$

retaining the first ten modes in expansion (12) we have

$$D_{\Delta S}(\rho) = \frac{576\langle a_7^2\rangle}{R^6}\,\rho^2 v^4\tau^4 , \quad (17)$$

where $\langle a_7^2\rangle = 0.00619(D/r_0)^{5/3}$.

Below we give the formulas, in which we use the following notation:

$$D_s(\rho_1 - \rho_2) = 6.88(|\rho_1 - \rho_2|/r_0)^{5/3} = 2.17(D/r_0)^{5/3}(\rho/R)^{5/3}$$

is the structure function of phase distortions introduced by the atmospheric turbulence,

$$D_s(\rho_1,\ \rho_2) = 1.796(D/r_0)^{5/3}\,|\rho_1 - \rho_2|^2/R^2 + 0.42(D/r_0)^{5/3} \times [(x_1^2 - x_2^2) - (y_1^2 - y_2^2)]^2/R^4 + 0.56(D/r_0)^{5/3}(x_1\, y_2 - x_2\, y_1)^2/R^4$$

is the structure function derived when we retain the six modes in the expansion,

$$D_s'(\rho_1 - \rho_2) = 1.11(D/r_0)^{5/3}(\nu\tau/R)^2(\rho/R)^2 \left[1 + \frac{\sin 2\theta}{2}\right]$$

is the structure function of residual phase distortions in the system with "constant delay",

$$D_s''(\rho_1, \rho_2) = 3.56(D/r_0)^{5/3}(\nu\tau/R)^4(\rho/R)^2 , \ \rho_1 \rho_2 = \rho^2\sin\theta\cos\theta$$

is the structure function of residual distortions of the "fast" adaptive system.

The results of numerical calculations based on these formulas are shown in Fig. 1 for the following values of the Strehl factors:

$$\mathrm{St} = \frac{\int_0^1 \mathrm{d}x x \tilde{K}(x) \exp\{- 3.44(D/r_0)^{5/3}x^{5/3}\}}{\int_0^1 \mathrm{d}x x \tilde{K}(x)} ; \quad (18)$$

$$\mathrm{St}'(\tau) = \int_0^\pi \mathrm{d}\theta \int_0^1 \mathrm{d}x x \tilde{K}(x) \exp\left[- 2.23(D/r_0)^{5/3} (\nu\tau/R)^2x^2 \times \right.$$
$$\left. \times \left(1 + \frac{\sin 2\theta}{2}\right)\right] \left[\pi \int_0^1 \mathrm{d}x x \tilde{K}(x))\right]^{-1} . \quad (19)$$

$$\mathrm{St}''(\tau) = \frac{\int_0^1 \mathrm{d}x x \tilde{K}(x) \exp[- 7.12(D/r_0)^{5/3} (\nu\tau/R)^4 x^2]}{\int_0^1 \mathrm{d}x x \tilde{K}(x)} ; \quad (20)$$

The calculated parameter here is $(D/r_0)^{5/3}$ while $\nu\tau/R$ is the generalized argument. The same figure shows the results of calculations based on formula (9). The values of $\nu\tau/R$ larger than St, can be easy found from these graphs. The graphs also show the advantage of the "fast" adaptation over the algorithm with "constant delay".

SPECTRAL CHARACTERISTICS OF ADAPTIVE SYSTEMS

Two–color adaptive systems are most often encountered in practice, since the wavelentgh in the reference channel and in the system of control differ from the corrected wavelength. Analitic relations were derived in Ref. 5 and the variance of difference between the characteristic functions and their structure functions was analyzed asymptotically. For example, the variance of the difference between the characteristic functions is

$$\sigma^2_{\Delta\theta} = 2\pi^2 \int_0^L \mathrm{d}x \int_0^\infty \mathrm{d}\kappa\kappa\, \Phi_n(\kappa, x) \left[\cos\frac{\kappa^2 x}{2k_1} - \cos\frac{\kappa^2 x}{2k_2}\right]^2 . \quad (21)$$

The Kolmogorov model of spectral density of function with a finite outer scale was used in our calculations

$$\Phi_n(\kappa, x) = 0.033C_n^2(x) \left(\kappa^2 + \kappa_0^2\right)^{-11/6} e^{-\kappa^2/\kappa_m^2} . \quad (22)$$

The vertical profile of the atmospheric turbulence was prescribed in the form

$$C_n^2(h) = C_n^2(0)\begin{cases} (1 + h/h_0)^{-2/3} , & h \le h_1 , \\ (h/h_0)^{-4/3}(1 + h_1/h_0) , & h > h_1 , \end{cases} \quad (23)$$

where $h \in [0, L]$. The following parameters were used in these calculations: outer scale $2\pi/\kappa_0$ = 100 m, inner scale $5.92/\kappa_m$ = 0.01 m, path length L = 1000 m, starting height h_0 = 30 m, height h_1 = 300 m, Fried's radius r_0 = 0.1 m at a wavelength of 0.55 μm, and diameter of reciving aperture D = 1 m.

The results of calculations are given in Table I. The first column gives the wavelength at which the correction was made, the second column – the wavelength of the corrected distortions. The rest columns give the corresponding variances and the functions related to them.

Numerical calculations were made for $C_n^2(0) = 4.93\cdot 10^{-15}\mathrm{m}^{-2/3}$. The corresponding equivalent heigths in model (23) appeared to be equal to

$$\int_0^L \mathrm{d}hC_n^2(h)/C_n^2(0) = 170\,\mathrm{m}, \ \int_0^L \mathrm{d}hhC_n^2(h) \Big/ \int_0^L C_n^2\, \mathrm{d}h = 271\,\mathrm{m}.$$

The third column gives the variance of the residual phase difference resulting from the "two–color" correction. The fourth column gives the values of the same variable normalized by $(D/r_0)^{5/3}$, the fifth – the standard deviation normalized to 2π rad. Finally, the sixth column gives the standard deviation measured in μm. For comparison note that the variance of phase fluctuations is 0.76 × 10^2 under the same conditions (at the wavelength of 0.5 μm)

TABLE I.

λ_1, μm	λ_2, μm	$\sigma^2_{\Delta s}\cdot 10^3$	$\sigma^2_{\Delta s}\cdot 10^5 / (D/r_0)^{5/3}$	$\sigma^2_{\Delta s}\cdot 10^3 / 2\pi$	$\sigma^2_{\Delta\theta}\cdot 10^3$, μm
0.50	1.0	3.12	5.55	8.88	4.44
0.50	2.0	10.3	18.3	16.1	8.06
0.50	3.0	17.0	30.2	20.7	10.4
0.50	4.0	23.4	41.6	24.3	12.2
0.50	5.0	29.5	52.6	27.4	13.7
0.50	6.0	35.5	63.3	30.0	15.0
0.50	7.0	41.4	73.6	32.4	16.2
0.50	8.0	47.1	83.8	34.5	17.3
0.50	9.0	52.7	93.8	36.5	18.3
0.50	10.0	58.2	104	38.4	19.2

The results of calculations given in Table I correspond to the outer scale of turbulence $2\pi/\kappa_0$ = 100 m. The calculations were also carried out for the model vertical profile $2\pi/\kappa_0 = 2\sqrt{h}$, and their results were practically identical to the data given in Table I. Thus, the variance of the difference between the characteristic functions at two different wavelengths is insensitive to the variations in the outer scale of turbulence. At the same time the variance of the difference between the characteristic functions $\sigma^2_{\Delta\theta}$ is quite sensitive to the changes in the inner scale of turbulence and to the path length. Table II shows the dependence of $\sigma^2_{\Delta\theta}$ on the changes in the inner scale of turbulence along the 1000–m path, and Table III illustrates such dependence on the path length for 0.01 m inner scale of turbulence.

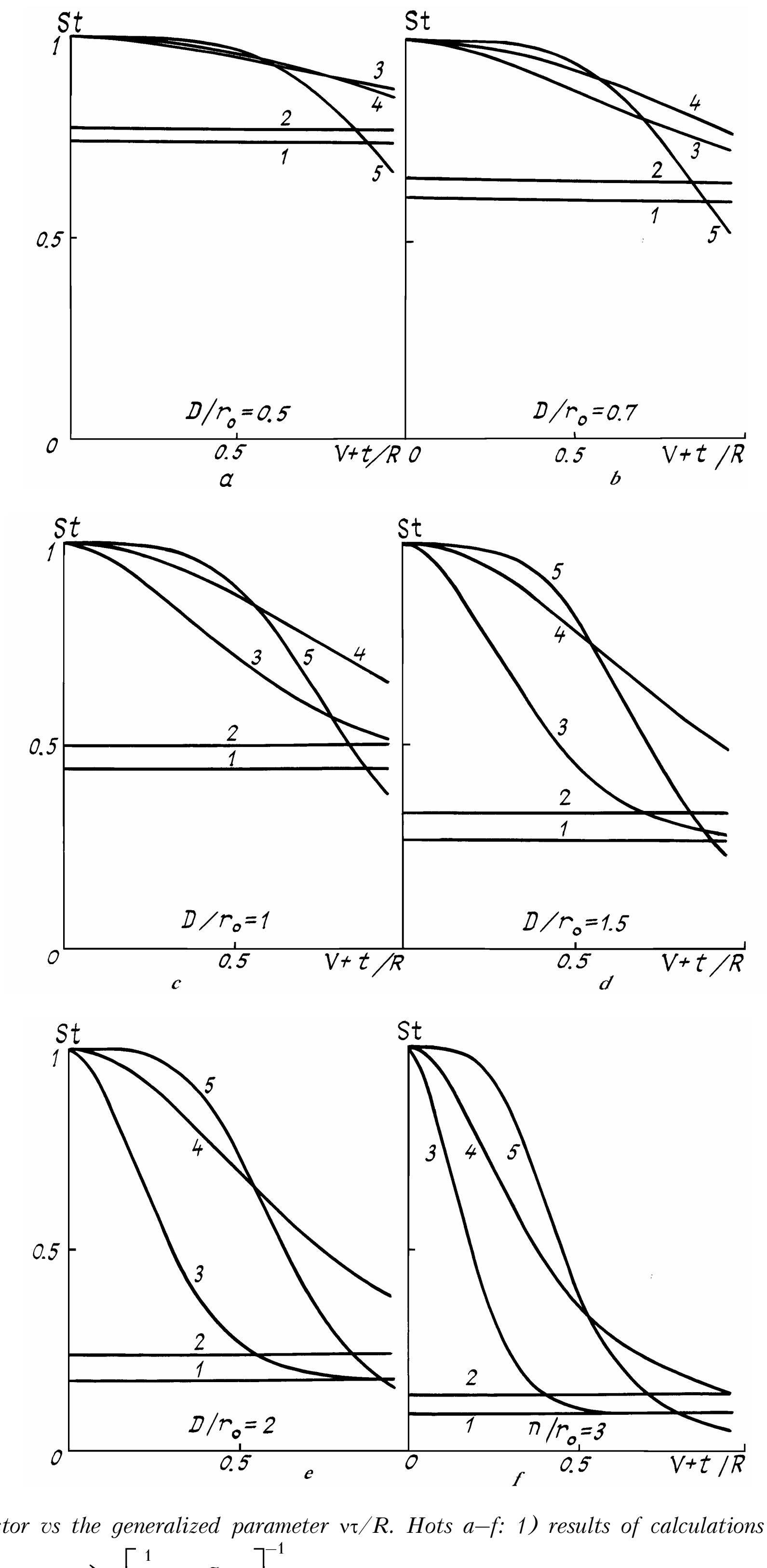

FIG. 1. The Strehl factor vs the generalized parameter $v\tau/R$. *Hots a–f: 1) results of calculations based on formula (18),* 2) $\int\limits_0^1 \mathrm{d}x\,x\,\tilde{K}(x)\exp\left(-\frac{1}{2}D_s(x)\right)\left[\int\limits_0^1 \mathrm{d}x\,x\,\tilde{K}(x)\right]^{-1}$, *3) results of calculations based on formula (9), 4) results of calculations of* $\mathrm{St}'(\tau)$ *based on formula (19), and 5) results of calculations of* $\mathrm{St}''(\tau)$ *based on formula (20).*

TABLE II.

Inner scale of turbulence $5.92\kappa_m$, m	0.03	0.01	0.003
Fluctuations of the optical path difference $\sigma^2_{\Delta\theta}$, μm	0.017	0.019	0.020

TABLE III.

Path length L, m	1000	3000	6000	10000
Fluctuations of the optical path difference $\sigma^2_{\Delta\theta}$, μm	0.0192	0.0256	0.0299	0.0346

Thus, the results of our calculations confirm once again the conclusions made in Refs. 5–7 and prove the effectiveness of "two–color" adaptive imaging. In addition, they show the possibilites of improving the dynamic characteristics of such systems.

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