LIGHT BEAM WITH AZIMUTH CARRIER FREQUENCY IN VACUUM AND IN AN INHOMOGENEOUS MEDIUM

V.A. Tartakovski and N.N. Mayer

Institute of Atmospheric Optics,
Siberian Branch of the Russian Academy of Sciences, Tomsk
Received July 31, 1998

We consider in this paper the problem on propagation of a Gaussian beam with a real zero in the initial conditions. We have derived a solution to the Helmholtz equation that describes this problem for vacuum. We have calculated numerically the probability of conservation of the azimuth carrier frequency, produced by the real zero, during the wave propagation through an inhomogeneous medium.

1. INTRODUCTION

Let a light wave propagates along the vertical coordinate axis and has real zero in a horizontal plane. Around this zero an azimuth oscillation exists, whose phase has the vortex shape. The size of a vortex is determined by the size of the zero-point neighborhood where the phase remains a monotonic function of the azimuth angle. The derivative of the linear component of the phase vortex with respect to the angle has the meaning of the azimuth carrier frequency, that has only integer values equaled to the order of the zero.

The propagation of a wave through an inhomogeneous medium results in modulation of the azimuth carrier frequency. The increasing turbulence will, in the final result, break the monotonic behavior of the phase around the zero point, and the azimuth carrier frequency disappears.

The phenomena described have been, in part, considered in Refs. 1, 2, and 3. In this paper we solve the problem on propagation of a Gaussian beam with the azimuth carrier frequency through vacuum in a very simple way, and study, by numerical simulations, the propagation of such a beam through turbulent media.

2. PROPAGATION OF A LIGHT BEAM WITH AZIMUTH CARRIER FREQUENCY THROUGH VACUUM

A monochromatic scalar field may be presented, in a cylindrical coordinate system r, ϑ, and z, in the following form:

$$W(r,\ \vartheta,\ z) = \sum_{m=-\infty}^{\infty} e^{im\vartheta} \int_0^{\infty} S_m(\rho,\ z)\ J_m(\rho r)\ \rho d\rho. \qquad (1)$$

In this expression the variable z shows the direction of wave propagation, the angle ϑ is the azimuth, m has the meaning of the azimuth frequency, ρ is the radial frequency, and $S_m(\rho, z)$ is the spatial spectrum in the plane z. The symbol J_m denotes the Bessel function of the first kind and of the order m.

The field (1) should obey the Helmholtz equation

$$\left(\frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} + \frac{1}{r^2}\frac{\partial^2}{\partial \vartheta^2} + \frac{\partial^2}{\partial z^2} + k^2\right) W(r, \vartheta, z) = 0. \qquad (2)$$

By substituting Eq. (1) into Eq. (2), we obtain the following expression:

$$0 = \sum_{m=-\infty}^{\infty} e^{im\vartheta} \int_0^{\infty} \rho \left\{ \left[\frac{\partial^2 S_m(\rho,z)}{\partial z^2} + k^2\ S_m(\rho,z)\right] J_m(\rho r) + \right.$$
$$\left. + \rho^2 \left[\frac{\partial^2 J_m(\rho r)}{\partial(\rho r)^2} + \frac{1}{\rho r}\frac{\partial J_m(\rho r)}{\partial(\rho r)} - \frac{m^2}{(\rho r)^2} J_m(\rho r)\right] S_m(\rho,z)\right\} d\rho. \qquad (3)$$

Taking into account, that the expression within the second square bracket in Eq. (3) is equal to $-J_m(\rho r)$, we may write the differential equation for the spatial spectrum in the following form:

$$\frac{\partial^2 S_m(\rho,\ z)}{\partial z^2} + (k^2 - \rho^2)\ S_m(\rho,\ z) = 0\ , \qquad (4)$$

and the partial solution of this equation, for the wave propagating in the positive direction along the z axis, with the initial condition $S_m(\rho)$ is as follows:

$$S_m(\rho,\ z) = S_m(\rho)\ \exp iz\ \sqrt{k^2 - \rho^2}\ . \qquad (5)$$

To make further calculations, we present the square root in this expression by the first terms of the power series, assuming the radial spatial frequency ρ to be small as compared to the wave number k

0235-6880/98/11 1001-05 \$02.00

$$\sqrt{k^2-\rho^2} = k\sqrt{1-\frac{\rho^2}{k^2}} \to k\left(1-2\,\frac{\rho^2}{k^2}\right) . \qquad (6)$$

Let the field in the plane of the source at $z = 0$ be Gaussian function with the vortex factor $r^n\, e^{im\vartheta}$, which is real zero of the order n, at the origin of coordinates:

$$W(r,\,\vartheta) = r^n e^{im\vartheta} \exp\left(-\frac{r^2}{2c^2} - i\,\frac{r^2k}{2R}\right) = W(r)\, e^{im\vartheta}. \qquad (7)$$

Here c and R are the constants characterizing the initial width of the beam and the curvature of the wave front.

The spectrum $S_m(\rho)$ in the source plane is determined by the following decomposition of the function $W(r)\, e^{im\vartheta}$:

$$S_m(\rho) = \int\limits_0^{2\pi} e^{i(n-m)\vartheta}\, d\vartheta \int\limits_0^{\infty} W(r)\, J_m(\rho r)\, r dr =$$

$$= \int\limits_0^{\infty} \delta_{mn}\, W(r)\, J_m(\rho r)\, r dr = S_n(\rho). \qquad (8)$$

By substituting the radial part of the field (7) in this decomposition and calculating the integral, we obtain the expression for spectrum in the plane of the source

$$S_n(\rho) = \int\limits_0^{\infty} r^{n+1}\, J_n(r\rho) \exp\left(-\frac{r^2}{2c^2} - i\,\frac{r^2k}{2R}\right) dr =$$

$$= \rho^n c^{2(n+1)} \left(1 + ic^2\,\frac{k}{R}\right)^{-(n+1)} \exp\left[-\frac{\rho^2c^2}{2}\left(1+ic^2\,\frac{k}{R}\right)\right]. \qquad (9)$$

In so doing, we used the integral tabulated in Ref. 4

$$\int\limits_0^{\infty} r^{n+1} \exp\,(-\,ar^2)\, J_n(\rho r)\, dr = \frac{\rho^n}{(2a)^{n+1}} \exp\left(-\frac{\rho^2}{4a}\right), \qquad (10)$$

$\mathrm{Re}\, a > 0$, $\mathrm{Re}\, n > -1$, $\rho > 0$.

Thus, for all m, except $m = n$, the spectrum $S_m(\rho,\,0)$ is equal to zero, and from the sum in Eq. (1) only the field remains that is caused by the spectrum $S_n(\rho,\,z)$, that allows one to find the field in the plane of the receiver using the approximation (6) and the integral (10)

$$W(r,\,\vartheta,\,z) = \int\limits_0^{\infty} \rho\, J_n(\rho r)\, S_n(\rho) \exp ikz\left(1-2\,\frac{\rho^2}{k^2}\right) d\rho =$$

$$= r^n\, e^{im\vartheta}\, e^{ikz}\left(1-\frac{z}{R}+iD\right)^{-(n+1)} \times$$

$$\times \exp \frac{-\,r^2\left(1 + i\,\dfrac{z}{RD}\right)}{2c^2\left(1+\dfrac{z}{R}+iD\right)}\,, \qquad (11)$$

where $D = z/kc^2$ is the dimensionless diffraction length.

It is seen, that the real zero is at the center of the beam at all z, the order of the zero does not vary at propagation, and the azimuth carrier frequency n also keeps present. This corresponds to the result obtained in Ref. 1. At $n = 0$ expression (11) reduces to usual Gaussian beam[5] without the vortex factor.

Note, that no analytical solution solely for the vortex phase factor $e^{im\vartheta}$ has been obtained. At numerical simulation of the beam propagation with this factor the artifacts arise in the form of high frequency amplitude and phase modulations. It is evidently connected with that the function $e^{im\vartheta}$ is undefined at the point $r = 0$, as it was noted in Ref. 2.

Let us study focal spots, that are produced by a Gaussian beam with the zero factor of a more general view, $x + (a + ib)\, y$, which is a combination of linear terms of a power series in the neighborhood of zero point. Let us set the beam in a Cartesian coordinate system in the following form:

$$W(x,\,y) = [x + (a + ib)y] \exp\left(-\frac{x^2+y^2}{2q^2}\right), \qquad (12)$$

where $q^2 = c^2\, R/(R + ikc^2)$ is the complex constant.

Using the relations

$$\exp\left(-\frac{x^2}{2q^2}\right) \xrightarrow{\ \mathbf{F}_{x\alpha}\ } \sqrt{2\pi}\ q \exp\left(-\frac{q^2\alpha^2}{2}\right),$$

$$x \exp\left(-\frac{x^2}{2q^2}\right) \xrightarrow{\ \mathbf{F}_{x\alpha}\ } i\sqrt{2\pi}\ q^3\alpha \exp\left(-\frac{q^2\alpha^2}{2}\right),$$

where $\mathbf{F}_{xa}$ is the operator of one-dimensional Fourier transform, we perform a two-dimensional Fourier transformation of the expression (12), in Cartesian coordinates, and find the representation for the spatial spectrum of a plane wave in a Gaussian beam with the real zero at its center

$$S(\alpha,\,\beta) = i\ 2\pi\, q^4\, [\alpha + (a + ib)\,\beta] \times$$

$$\times \exp\,[-\,(\alpha^2 + \beta^2)\, q^2/2]. \qquad (13)$$

Let the focal spot be the square of the module of this spectrum

$$|S(\alpha,\,\beta)|^2 = (2\pi)^2 |q|^8\, [\alpha^2 + (a^2 + b^2) \times$$

$$\times\, \beta^2 + 2a\, \alpha\beta] \exp\,[-\,|q|^8\, (\alpha^2 + \beta^2)/c^2], \qquad (14)$$

where a, b, and c are real-valued constants.

The spatial spectrum (13) of the Gaussian beam with the vortex factor has the same factor in the frequency range $\alpha\beta$. From that it follows, in particular, that both these functions have no a constant component. Because of the central symmetry of function $|S(\alpha,\,\beta)|^2$, its axial moments of inertia of the first order will be equal to zero. Therefore, the field has no a common tilt of the wave front in a small neighborhood of the real zero point.

The expression for the focal spot in Eq. (14) contains, as a constant factor, the second order surface in the coordinate system $\alpha\beta$ in the square brackets, that explains the dichotomy of the focal spots[2,3] and also their ring shape (Fig. 1, frames *3*, *5*, and *7*).

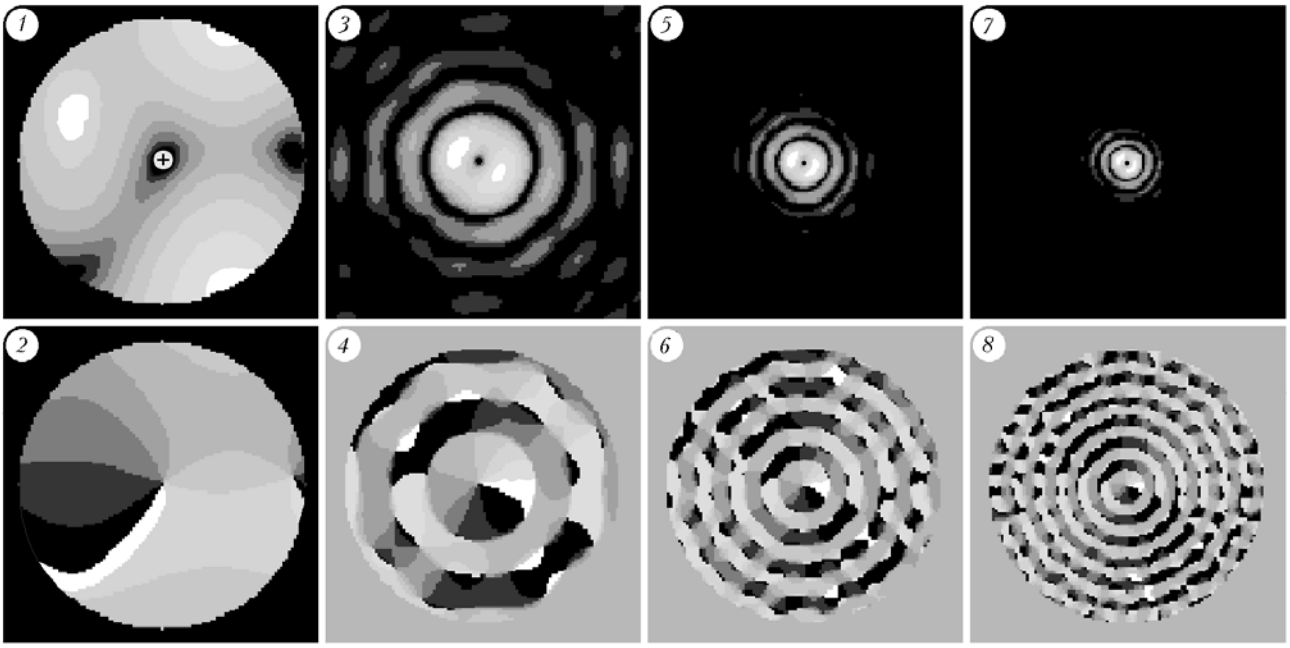

FIG. 1. Real-zeros and the phase vortices of a wave and its Fourier transforms when propagating through an inhomogeneous medium ($C_n^2 = 10^{-14}$ m$^{-2/3}$, L = 2000 m). Intensity of the wave with the real zero at the center of the Hartmann subaperture (+) (1) and the corresponding phase (2). Focal spots (3, 5, and 7) and the phase (4, 6, and 8), which correspond to the focal spots, located opposite to the wave with the real zero (1 and 2) when increasing the size of the subaperture.

3. PROPAGATION OF A LIGHT BEAM WITH THE AZIMUTH CARRIER FREQUENCY THROUGH AN INHOMOGENEOUS MEDIUM

We have carried out a numerical experiment to study the phase of the beam with the vortex factor propagated through an inhomogeneous medium. We used known numerical model described in Ref. 6. The Gaussian beam and its spatial spectrum were approximated by periodic functions and then entered into a computer as two-dimensional matrices of readouts. Thus formed matrix has the dimensionality 100, that ensures the adequacy of representing continuous functions by their discrete prototypes. We have used two phase screens in our simulation of the inhomogeneous medium with the spectral density of the refractive index corresponding to the atmospheric turbulence in the inertial interval. The law of energy conservation holds in the model with the computer accuracy. The spectral density of the phase fluctuations and other model parameters are defined by the following relations:

$$F_s(\varkappa) = 0.489\ r_o^{-5/3}\ (\varkappa^2 + \varkappa_o^2)^{-11/6},$$
$$\varkappa_o = \frac{2\pi}{L_o},\ r_o = \left\{0.423\ k^2 \int\limits_L C_n^2(l)\ \mathrm{d}l\right\}^{-3/5}, \qquad (15)$$

where r_o is the Fried radius of coherence, C_n^2 is the structure constant of the refractive index, $\varkappa$ is the spatial frequency, L is the path length. The outer scale of the turbulence L_o is equal to 1 m, $k = 2\pi/\lambda$, the wavelength λ is equal to 0.6328 μm.

It is seen from Fig. 1, that the wave, propagated through an inhomogeneous medium ($C_n^2 = 10^{-14}$ m$^{-2/3}$, L = 2000 m) and having the real zero within the Hartmann sub-aperture without screening, forms the focal spots with zero as well. Both functions have the azimuth carrier frequency in the neighborhood of the zero point. This property is not influenced by the sub-aperture size, at least when the zero point is inside the sub-aperture, and its size is less than the size of the phase vortex.

The vortex size can be estimated by the diameter of the greatest circle centered at the zero point, where the phase is still a monotonic function of the azimuth angle. In the plane z = const in the neighborhood of the zero point, the phase both of the wave functions (11) and (12) and the spatial spectrum (13) is the phase of vortex factor $x + (a + ib)\ y$. In the coordinate system $r\,\vartheta$ it does not depend on r and has the form

$$\varphi_\circ(\vartheta) = \arctan \frac{b \sin\vartheta}{\cos\vartheta + a \sin\vartheta}. \qquad (16)$$

Let us find the derivative of this phase with respect to the azimuth angle ϑ

$$\frac{d\varphi_\circ}{d\vartheta} = \frac{b}{(b\sin\vartheta)^2 + (\cos\vartheta + a\sin\vartheta)^2}. \quad (17)$$

From this expression it is seen that the derivative does not change the sign with varying ϑ. Consequently, the phase of the wave in the neighborhood of the real zero point is a monotonic function of the azimuth angle with the vertex at the real-zero point.

Thus, the presence of simultaneously the real zero, the azimuth carrier frequency in the wave and in its spectrum of plane waves takes place both in vacuum, and in an inhomogeneous medium.

We defined in numerical experiment the path length in an inhomogeneous medium, along which the azimuth carrier frequency of a Gaussian beam with the vortex factor in the form (7) and (11) conserves. The presence of the azimuth carrier frequency in the beam during its propagation through an inhomogeneous medium was set visually by the presence or absence of the monotonic function of phase around the zero point. We have considered two variants: the presence of a monotonic phase in the ring area, where the intensity of the wave is large, and in a small neighborhood of the zero point, where the intensity is small. The probability that the azimuth carrier frequency conserves was determined by the ratio of the number of favorable cases to the sample size equaled to 10. The samples of the real part of the wave, its intensity, and phase for different orders of the carrier frequency are shown in Fig. 2. The results are also presented in Table I.

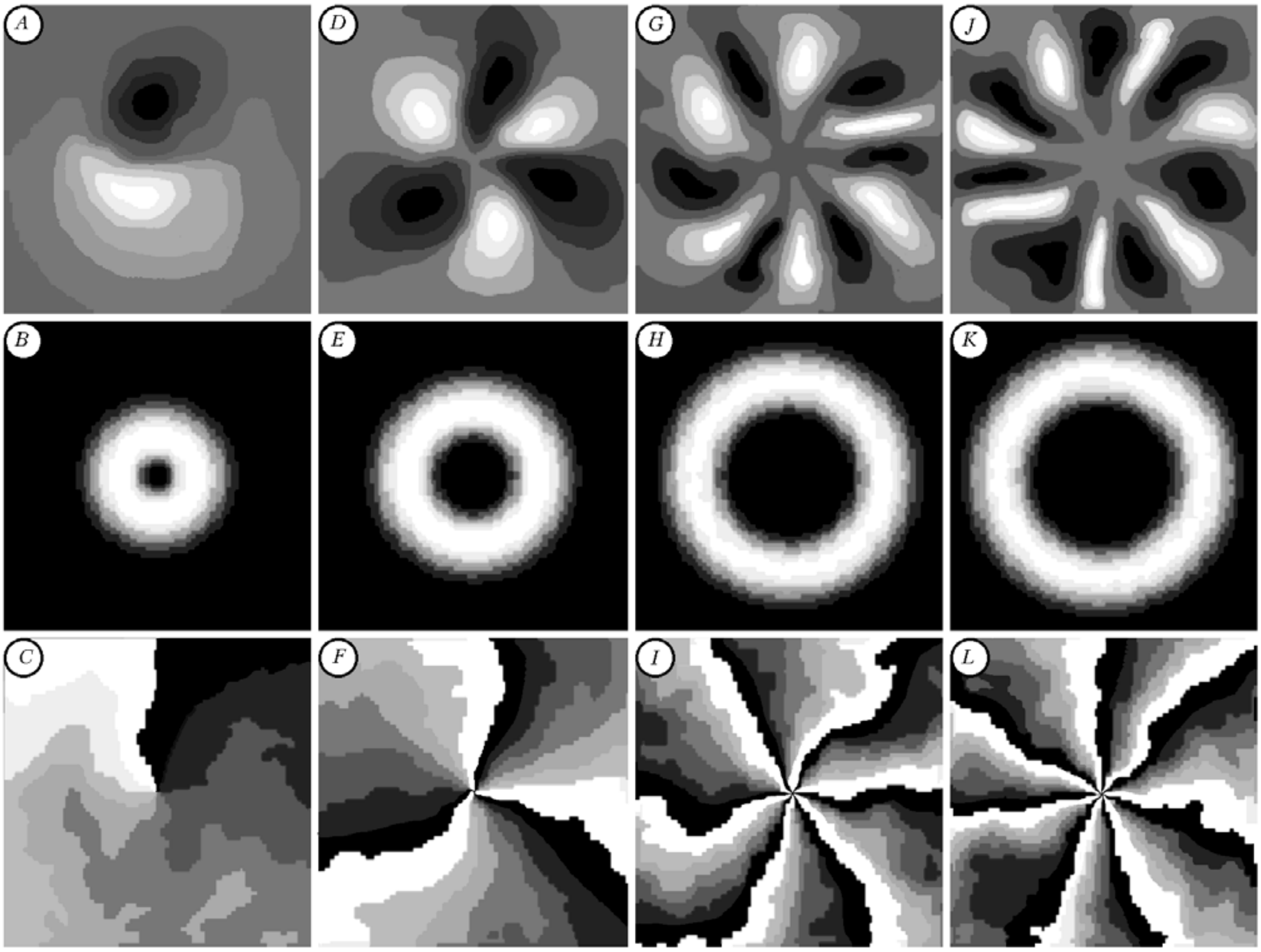

FIG. 2. The beam with the azimuth carrier frequency in an inhomogeneous medium. Each column corresponds to one realization. Upper row shows the real part of the wave, the middle one corresponds to the intensity, and the lower one to the phase. The order of the azimuth carrier is equal to unity for the first column, three – for the second one, six – for the third one, and seven – for the fourth column. $C_n^2 = 10^{-15}$ m$^{-2/3}$, $r_\text{o} = 0.74$ m, L=100 m, *and* $\beta_\text{o}^2 = 0.0005$ *for the first and second columns.* $C_n^2 = 10^{-15}$ m$^{-2/3}$, $r_\text{o} = 0.27$ m, $L = 500$ m, *and* $\beta_\text{o}^2 = 0.009$ *for the third and fourth columns. The size of the matrix is equal to* 1 m.

TABLE I. The sampling probability p_c of the n-order azimuth carrier conservation versus the path length L and turbulent characteristics C_n^2, r_o, and β_o^2.

r_o, m	C_n^2, $m^{-2/3}$	β_o^2	L, m	1	2	3	4	5	6	7	8	9	10	n
0.74	10^{-15}	0.0005	100	1	1	1	1	1	1	1	1	1	1	
0.27	10^{-15}	0.0090	500	**0.3**	0.6	0.6	1	1	1	1	1	1	1	
0.18	10^{-15}	0.0330	1000	**0.2**	0.3	0.4	0.6	0.8	1	1	1	1	1	
0.14	10^{-15}	0.0690	1500	**0.2**	0.2	0.4	0.5	0.9	1	1	1	1	1	p_c
0.12	10^{-15}	0.1200	2000	**0.2**	**0.9**	0.4	0.4	0.6	0.8	1	1	1	1	
0.18	10^{-14}	0.0050	100	**0.6**	**1**	0.3	0.7	1	1	1	1	1	1	
0.70	10^{-14}	0.0920	500	**0.2**	**0.2**	**0.6**	**0.9**	**0.9**	0.2	0.3	0.8	0.6	0.7	

Note: The bold-type values of the probability correspond to the carrier frequency in small neighborhood around real-zero point, the other values of probability correspond to the presence of the carrier in the ring area (see Fig. 2, frames B, E, H, and K). The sample size is equal to 10.

Analysis of data compiled in this table shows, that the azimuth carrier frequency of a higher order conserves better at propagation in an inhomogeneous medium. It is quite clear, because the higher the carrier frequency, the wider spectrum can have a modulating oscillation without any loss of properties of an analytic signal.[7] The higher stability of the carrier in the central part of a beam rather than in the ring area may be explained by the power spectrum of the inhomogeneities in the refractive index of medium, what results in a stronger influence of large-scale inhomogeneities on the wave phase fluctuations.

4. CONCLUSION

The azimuth carrier frequency, given by the initial conditions, conserves at propagation of a light beam in vacuum.

The higher the order of the azimuth carrier, the longer the path length where it conserves at propagation through an inhomogeneous medium.

The azimuth carrier is present at a focal plane or in the spatial frequency domain for beams propagated both through vacuum and an inhomogeneous medium.

The form of the focal spot from Hartmann subaperture with the real zero is determined mainly by first terms of a two-dimensional power series of the wave function in the neighborhood of this zero point.

REFERENCES

1. E. Abramochkin and V. Volostnikov, Opt. Commun. **102**, Nos. 3, 4, 336–350 (1993).
2. V.P. Lukin and B.V. Fortes, Atmos. Oceanic Opt. **8**, No. 9, 223–230 (1995).
3. V.A. Tartakovsky and N.N. Mayer, Atmos. Oceanic Opt. **9**, No. 11, 926–929 (1996).
4. I.S. Gradshtein and I.M. Ryzhik, *Tables of Integrals, Sums, Series, and Products*, 6.631(4).
5. M.B. Vinogradova, O.B. Rudenko, and A.P. Sukhorukov, *The Wave Theory* (Nauka, Moscow, 1979), 384 pp.
6. V.P. Lukin, N.N. Mayer, and B.V. Fortes, Atm. Opt. **4**, No. 12, 896–899 (1991).
7. V.A. Tartakovsky, Atmos. Oceanic Opt. **10**, No. 3, 189–197 (1997).