RYTOV APPROXIMATION: COMMENTS CONCERNING THE RANGE OF ITS APPLICABILITY

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We consider here some aspects of the problem on applicability limits of Rytov approximation that is used for calculating turbulence-induced fluctuations. The errors of approximation are estimated by way of comparing the results calculated by Rytov with those obtained by numerically solving the parabolic equation. It is shown that the applicability range of interest strongly depends on the type of quantities to be calculated. Also, some aspects related to the singular behavior of the amplitude in the presence of phase dislocations are discussed.

The Rytov's approximate solution of the parabolic equation[1] is a common approach used to calculate light wave fluctuations induced by weak atmospheric turbulence. It is normally assumed that this solution is valid while the so-called Rytov scintillation index $\beta_0^2 < 0.3 - 1$, Refs. 1–3. However, the applicability domain cannot be assessed correctly without the account of the type of a quantity to be calculated using this approximation. This means that, under same conditions of propagation and turbulence, the errors of approximation can be different for different quantities. The physical mechanism which affects the applicability domain of interest, can be considered as follows. The main advantage of Rytov's approach is that it accounts for the effect of multiple scattering. However, it allows for this effect only partially and the prediction by Rytov's approximation becomes wrong at the enhanced level of multiple scattering. So, one can expect that the applicability of Rytov's approach to handling certain quantities is actually dependent on how strongly the multiple scattering affects the quantity of interest. In other words, the stronger some quantity is affected by the multiple scattering, the narrower is the applicability range of Rytov's approach for this quantity. Below we outline this problem by comparing the results obtained by Rytov's approximation with those obtained from the simulation based on the numerical solution of the parabolic equation. Three statistical quantities are compared: the variance of the logarithm of amplitude, the variance of its first derivative, and the variance of its second derivative. These quantities are chosen for the following reasons. On the one hand, the small-scale atmospheric inhomogeneities produce the main contribution into the multiple scattering. At the same time, these inhomogeneities are mainly responsible for the behavior of the derivatives of the logarithm of amplitude. So, one can expect that, under the same propagation and turbulence conditions, the error of Rytov approximation increases with increasing order of the derivative.

We restrict below our attention to the case of a plane monochromatic wave propagating through the turbulent atmosphere along a path with constant parameters. The refractive index fluctuations are assumed to be Gaussian and isotropic with the power spectrum Φ_n being as follows:

$$\Phi_n(\varkappa) = 0.033\, C_n^2\, \varkappa^{-11/3} \exp(-\varkappa^2/\varkappa_m^2),$$
$$\varkappa_m = 5.92/l_0, \qquad (1)$$

where C_n^2 is the refractive-index structure constant, and l_0 denotes the inner scale of turbulence.

The expressions proposed by Rytov for the variances of interest are:

the variance $\langle\varkappa^2\rangle$ of the logarithm of amplitude

$$\langle\varkappa^2\rangle = 2.175\, C_n^2\, k^2\, L\, \varkappa_m^{-5/3} \times$$
$$\times \left\{-1 + \frac{6}{11} D^{5/6} \left(1 + \frac{1}{D^2}\right)^{11/12} \sin\left(\frac{11}{6} \arctan D\right)\right\}, \quad (2)$$

the variance $\langle\varkappa_x^2\rangle$ of the first derivative of the logarithm of amplitude

$$\langle\varkappa_x^2\rangle = 0.907\, C_n^2\, k^2\, L\, \varkappa_m^{1/3} \times$$
$$\times \left\{1 - \frac{6}{5} D^{-1/6} \left(1 + \frac{1}{D^2}\right)^{5/12} \sin\left(\frac{5}{6} \arctan D\right)\right\}, \quad (3)$$

the variance $\langle\varkappa_{xx}^2\rangle$ of the second derivative of the logarithm of amplitude

$$\langle\varkappa_{xx}^2\rangle = 0.113\, C_n^2\, k^2\, L\, \varkappa_m^{7/3} \times$$
$$\times \left\{1 - 6\, D^{-7/6} \left(1 + \frac{1}{D^2}\right)^{-1/12} \sin\left(\frac{1}{6} \arctan D\right)\right\}, \quad (4)$$

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where L is the propagation path length, k is the wave number, and $D = \frac{L\varkappa_m^2}{k}$ is the wave parameter.

The Eqs. (2)–(4) directly follow from the general expression for the correlation function of the logarithm of amplitude.[1]

To estimate the errors of Rytov's approximation for the above variances, we compare the approximate theoretical results (1)–(4) to those obtained by numerically solving the parabolic equation[1]

$$2ik\,\frac{\partial E(z,\rho)}{\partial z} + \Delta_{\perp}E(z,\rho) + 2k^2\,\tilde{n}(z,\rho)\,E(z,\rho) = 0, \quad (5)$$

where E denotes the complex wave field, $\rho = (x, y)$ is the vector defined in a plane perpendicular to the direction of propagation, $\Delta_{\perp} = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}$, and $\tilde{n}$ denotes the fluctuation component of the refractive index.

Our method of simulation is similar to that presented in Ref. 4, so we do not describe it here referring the reader to this paper for details. Figure 1 depicts the relative errors of approximation (in percent) for the variances of interest versus scintillation index β_0^2 given by Rytov's formula

$$\beta_0^2 = 1.23\ C_n^2\ k^{7/6}\ L^{11/6}.$$

For each variance σ^2, the relative error δ is calculated as

$$\delta = 100\%\ |\ \sigma_R^2 - \sigma_S^2\ |\ / \sigma_R^2\ ,$$

where σ_R^2 and σ_S^2 denote the Rytov's and calculated magnitude of a given variance, respectively. The variances have been calculated for a sample of 100 values.

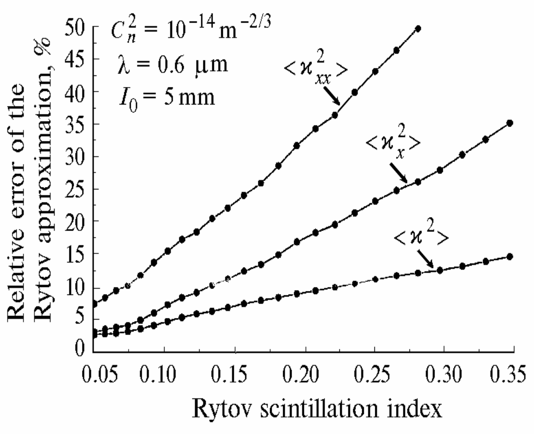

FIG. 1. Relative errors of Rytov's approximation for three quantities: the variance $<\varkappa^2>$ of the logarithm of amplitude, the variance $<\varkappa_x^2>$ of its first derivative, and the variance $<\varkappa_{xx}^2>$ of its second derivative.

As one can see from Fig. 1, the behavior of the curves presented confirms our initial assumptions. Besides, another one, and more general, conception of the applicability region of Rytov's approximation may be proposed. As was mentioned above, the multiple scattering which is only partially allowed for within this approximation is mainly produced by small-scale inhomogeneities. On the other hand, the inner scale is a quantity that actually determines the contribution coming from small-scale inhomogeneities into the refractive-index spectrum. Thus, we may assume that the stronger is the dependence on the inner scale of some quantity calculated using Rytov's approximation, the narrower is the applicability limits of this result.

In the above discussion we have presented the results of simulation made assuming conditions of weak-turbulence. Under these conditions the variances of interest vary smoothly with the increase in Rytov's scintillation index or, in other words, with the turbulence strengthening. However, as soon as the conditions start to approach to the strong-turbulence ones, one can observe a qualitatively different effect: a singular behavior of the variances of derivatives of the logarithm of amplitude related to the appearance of phase dislocations (vortices).[5–10]

It is known[6] that a necessary condition of phase vortex creation at some observation point is the occurrence of zero amplitude at this point. Up to now the main attention in the problem of dislocations has been paid to the phase singularities, while the statistical and topological properties of the amplitude in the presence of zero-amplitude points have remained practically beyond the scope of consideration. However, not only the phase, but also the amplitude demonstrates some unusual properties when the zero-amplitude points occur, namely, the amplitude derivatives are singular at these points too.

Let $E(x, y) = E_1(x, y) + iE_2(x, y)$ be a sample of the complex two-dimensional wave field and let us assume that the amplitude zero occurs at some point inside this sample. Introducing the Cartesian system of coordinates with the origin at this point, and expanding E_1 and E_2 into a series in the vicinity of the origin, and taking only linear terms, one can obtain the following expression for the amplitude A:

$$A(x, y) = \sqrt{(E_{1x}x + E_{1y}y)^2 + (E_{2x}x + E_{2y}y)^2}, \quad (6)$$

where E_{1x}, E_{2x}, E_{1y}, and E_{2y} denote the corresponding first partial derivatives of the field taken at the origin.

As one can see from Eq. (6), the first derivative of the amplitude has a discontinuity at the origin, while the higher-order derivatives are singular. One can say that the amplitude rather "cuts" than "touches down" the zero plane at the point $x = 0$, $y = 0$.

To support the latter conclusion, let us consider how the presence of zero-amplitude point affects the variance $<\varkappa_{xx}^2>$ of the second derivative of the logarithm of amplitude and how this effect manifests itself in simulations. We are going to show that as soon as the first vortex appears inside the observation zone, the variance $<\varkappa_{xx}^2>$ tends to the infinity. To make further analysis simpler, let us consider the simplest case when $E_{1x} = E_{2y} = 1$ and $E_{2x} = E_{1y} = 0$ (a more general treatment is quite straightforward but it is not necessary for our consideration which is rather qualitative). By introducing the polar coordinates (ρ, φ) and using Eq. (7), we can present $\varkappa_{xx}^2(\rho, \varphi)$, in the vicinity of zero-amplitude point, as follows

$$\varkappa_{xx}^2(\rho, \varphi) = \frac{\cos^2(2\varphi)}{\rho^4} . \qquad (7)$$

A contribution σ_0^2 coming from the singularity (7) to the total variance $<\varkappa_{xx}^2>$ can be estimated as follows. Let us plot a circular zone with the center at the zero-amplitude point $\rho = 0$. Then, let ε and r be the inner and the outer radii of this ring, respectively. We can always choose such a small but finite r that the expansion (6) is valid inside the ring. Using Eq. (7), one can write the contribution σ_0^2 associated with this zone as

$$\sigma_0^2 = \lim_{\varepsilon \to 0} \frac{1}{\pi\,(r^2 - \varepsilon^2)} \int_\varepsilon^r \frac{d\rho\rho}{\rho^4} \int_0^{2\pi} d\varphi \cos^2(2\varphi) =$$

$$= \lim_{\varepsilon \to 0} \frac{1}{2r^2\,\varepsilon^2} = \infty. \qquad (8)$$

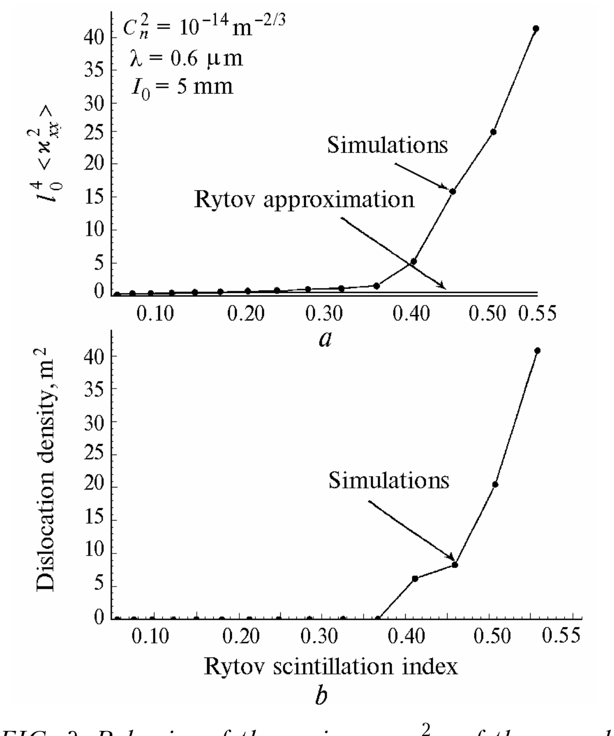

FIG. 2. Behavior of the variance $<\varkappa_{xx}^2>$ of the second derivative of the logarithm of amplitude in the presence of phase dislocations. The top graph panel presents the value $<\varkappa_{xx}^2>$ multiplied by the l_0^4 (the forth degree of the inner scale), while the bottom one shows the dislocation density.

So, when the wave field has a zero-amplitude point, the total variance $<\varkappa_{xx}^2>$ tends to the infinity due to the contribution that comes from the vicinity of this point. Figure 2 shows how this effect manifests itself in simulations (the method of simulation is described in Ref. 11). One can see a sharp growth of $<\varkappa_{xx}^2>$ as the dislocation density starts to be non-zero. In contrast to theoretical predictions, the simulated $<\varkappa_{xx}^2>$ does not tend to the infinity due to the finite grid step used in the simulation.

The above considered effect can be of certain importance when developing the theories of strong turbulence. As was shown in Ref. 11, the dislocations (or the zero-amplitude points) always occur under conditions of strong turbulence. This result, along with the above considerations, allows us to conclude that under conditions of interest the amplitude derivatives become singular. So, developing the theories of strong turbulence this effect should be taken into account.

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