RECONSTRUCTION OF THE INTERNAL STRUCTURE OF STRONGLY ABSORBING MEDIA FROM THE DATA ON THEIR EXTINCTION PROPERTY

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Mechanism of light propagation in strongly absorbing and inhomogeneous media is interpreted in terms of amplitude trajectories of the minimum extinction, the description of which is similar to that of the refraction in geometric optics. Under the assumption of axial symmetry of the profile of linear absorption coefficient, and using the inverse Abel transformation one may obtain the exact solution to the inverse problem on reconstruction of the absorption coefficient profile from the extinction measurement data. Using imitation simulations, the method is shown to work quite well for different profiles in the presence of measurement noise.

1. INTRODUCTION

Among many methods of solving inverse problems of the environmental sensing, of particular importance are the exact using explicit representation of the solution, for example in the form of integral transformations. As a rule, such solutions exhibit either high stability with respect to measurement noise, or enable one to assess an ultimate effect of the noise and to efficiently regularize them. Unfortunately, there are only few exact solutions known. Of those, most familiar is the solution of Abel equation to which the problems on direct light transmission through axially symmetric media are reduced, when the light source and receiver are outside the medium sounded onto which a beam is incident then suffering a refraction bending in it. In one or another way, the value of this bending is measured as a function of impact parameter. The radial structure of the refractive index is reconstructed by inverting Abel equation. Such a technique, pioneered in Ref. 1, was successfully used to study atmospheres of the Venus, Mars, Earth, and the Sun.[2] Normally the absorbing properties of the media are neglected when calculating the ray trajectories. In the case of strongly-absorbing media, the commonly used representation of the ray as the phase trajectory of a wave is senseless, thus making the refraction approach inapplicable directly.[3]

Recently, the methods of calculational tomography allowing for the effects of the radiation attenuation by substance has become widely used in sensing inhomogeneous media. The basic parameter reconstructed is the substance density, while the measured parameter is the attenuation of radiation intensity transmitted through the volume under study. Problem solution reduces to processing of multiple projections using inversion according to Radon transformation.[4,5] For axially symmetric media, this is shown to be equivalent to solving Abel equation.[5,6] In contrast to the case with the refraction, the wave trajectories are assumed to be straight lines. Wave extinction is considered to be proportional to the integral of substance density over these straight lines. Such an approach is applicable to weakly absorbing media; while in a more general case, one has to use a weighed integral density allowing for the background radiation attenuation by introducing an exponential factor.[4,5] It is just the inversion, in one or another way, of the multiview projections that makes the basis for the majority of tomographs operated.

In the case of strongly inhomogeneous media with a strong linear absorption it is hardly correct to assume the light propagation trajectories to be straight lines, since even the concept of a ray itself needs a revision. For the tomographic techniques in application to strongly absorbing media to be improved, not only more powerful radiation sources are needed, but also the mathematical apparatus for fluid structure reconstruction must be revised. Of many possible approaches, methods admitting solution in a closed form should be preferred.

The present paper makes a step in this direction by introducing the notion of wave amplitude trajectory. By the wave amplitude trajectory we mean a virtual curve along which the wave suffers the least attenuation. For axially symmetric media, the description of wave attenuation along this curve is analogous to the description of refraction in optics. This enables one to reduce the problem of reconstruction of the profile of the extinction coefficient from attenuation of wave to solving Abel equation, thus making use of known exact solution.

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2. AMPLITUDE WAVE TRAJECTORY IN STRONGLY ABSORBING MEDIUM

Contemporary interpretation of the ray trajectories relies on Huygens–Kirchhoff principle, i.e., on the concept of interference of wave perturbations, incurred by the initial wave at the source point and travelling to the observation point along all possible virtual trajectories.[3] For such an interference of secondary waves, significant contributors to the resulting field at the observation point are those trajectories the phase shift along which differs from the extreme value by no more than π. Spatial domain satisfying this condition is called significance zone of wave propagation (Fresnel zone) since the contribution coming from other trajectories is negligible. This conclusion is based on the estimate of the integral of full field by the method of steepest descents reducing to the method of stationary phase under the assumption of small wavelength $\lambda \to 0$. In that case, the Fresnel zone contracts into the so-called ray line. Such a definition is acceptable for weakly absorbing media. In the case when absorption cannot be neglected, estimate of the integral of full field by this same technique reduces to Laplace technique.[7] Significant contributors to the resulting field at the observation point are the trajectories on which the attenuation differs from its extreme value by no more than e times. The stronger attenuation, the closer the spatial domain, containing corresponding virtual trajectories, contracts around the extremal line. In that case, the propagation-significance zone can be identified with the wave amplitude trajectory.

Several words should be said about the approach applicability. First of all, we note that the linear absorption coefficient n is related to the imaginary part of the complex wave number as $n = \mathrm{Im}(2\pi\sqrt{\varepsilon}/\lambda_0)$, where λ_0 is the wavelength of radiation in free space, while $\varepsilon = \varepsilon_1 + i\varepsilon_2$ is the complex dielectric constant of the medium. Obviously, the concept of the amplitude trajectory proposed is alternative to the phase trajectory concept in the method of geometric optics. It can be used when the attenuation is large within several ($m \geq 3$–5) first Fresnel zones, whose size is estimated as

$$l \approx m\pi / \mathrm{Re}\,(2\pi\sqrt{\varepsilon}/\lambda_0)\,.$$

Taking this into account, the notion of amplitude trajectory is applicable under the condition that

$$m\pi\,\mathrm{Im}\,(\sqrt{\varepsilon_1 + i\,\varepsilon_2}) / \mathrm{Re}(\sqrt{\varepsilon_1 + i\,\varepsilon_2}) \gg 1.$$

This corresponds to the condition of strong absorption in the medium.

The wave attenuation along the amplitude trajectory is estimated as

$$\Gamma = \exp(-L),\;\; L = \int n\,\mathrm{d}l,$$

where the integral has extremely small value. For source and receiver, both fixed in space, this is equivalent to applying Ferma variational principle.[3] In other way, the expression for Γ can be derived immediately from the radiative transfer equation. It is important, that this integral coincides by form, to a constant factor, with the wave phase shift in the case of geometric optics, if n is meant to be refractive index.

Note that the concept of amplitude trajectory, as it is presented above, has been introduced mostly based on physical considerations rather than on rigorous estimates, thus being the euristic concept.

3. SNELL LAW AND ABEL EQUATION FOR STRONGLY ABSORBING MEDIA

Let us assume, for certain, that an inhomogeneous medium, viz absorbing object (AO), has spherically or cylindrically symmetric inner structure of the absorption coefficient $n(r)$, while the light source and receiver are located on the surface of known absorption level (Fig. 1). Then, in accordance to Ferma principle, the amplitude trajectory (curve *2*) of the wave propagation is found from the equation[3]

$$n(r)\; r \sin\alpha(r) = n(r_0)\; r_0 \sin\alpha(r_0) \equiv p,$$

being an analog of Snell law in geometric optics. Here $\alpha(r)$ is the angle of the trajectory inclination at a distance r from the center of symmetry; r_0 is AO radius; and p is the impact parameter of the trajectory. Amplitude trajectory does not coincide with the straight line connecting the source and receiver (line *1*). As a result, for attenuation along the optimal trajectory we have

$$L(p) = 2\int\limits_{r_{\min}}^{r_0} \frac{(n(r))^2\; r\,\mathrm{d}r}{\sqrt{[n(r)\; r]^2 - p^2}}\,, \qquad (1)$$

where $r_{\min}$ is the radius of the trajectory bending point, $r_{\min} n(r_{\min}) = p$. The angular distance between the entry and exit points of the optimal trajectory is[3]

$$\psi(p) = 2\int\limits_{r_{\min}}^{r_0} \frac{p\,\mathrm{d}r}{r\sqrt{[n(r)\; r]^2 - p^2}}\,. \qquad (2)$$

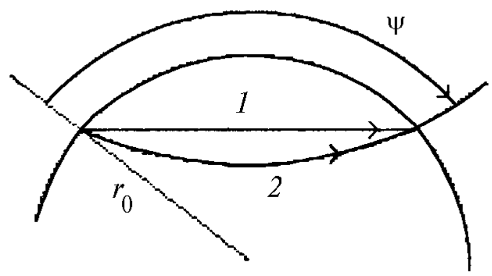

FIG. 1.

The value available from measurement is L, while the function to be sought is the radial profile $n(r)$. Once the attenuation is known as a function of impact parameter p from measurement, the problem is solved using inverse Abel transformation.[1,2] If the impact parameter is unknown and only the dependence $L = L(\psi)$ is recorded, the former one must be retrieved in some way in order to make use of Abel transformation.

As analysis showed, the impact parameter can be estimated as $p = \mathrm{d}L/\mathrm{d}\psi$, what can easily be shown by differentiating (1) and (2). With the account for the well-known solution of Abel equation, the problem we formulated is solved by evaluating the integral

$$\ln\left(\frac{r}{r_0}\right) = \frac{1}{\pi}\int\limits_{nr}^{n(r_0)\,r_0} \mathrm{d}\psi \, \ln\left\{\frac{p + \sqrt{p^2 - [nr]^2}}{nr}\right\}. \qquad (3)$$

From this, for a given value of the parameter nr we obtain corresponding r value, which finally enables the reconstruction of the dependence $n = n(r)$ sought.

4. IMITATIONAL SIMULATIONS

Results of imitational simulation confirmed high efficiency and stability of the solution proposed. Figure 2 presents examples of reconstruction of two different model profiles of the extinction coefficient (solid curves). Profiles of the extinction coefficient were fitted by the dependence

$$n(r) = n_0\,[1 + ar + b\,\sin(c\,(r - r_1))]\,\exp\,\{-\,d\,|r - r_1|\},$$

whose parameters can be adjusted to fit a wide range of profiles. The profiles shown in Fig. 2 were calculated for $a = 0.1$, $b = 0.5$, $n_0 r_0 = 2$, but curve *1* for $c = 1$, $d = 0.6$, $r_1 = 0.9r_0$, and curve *2* for $c = 0.5$, $d = 0$, $r_1 = 0.5r_0$. The direct problem was solved using simple numerical integration in (1) and (2); while the inverse problem was solved by a sequential numerical calculations by formula (3). The values n reconstructed are marked with crosses. Figure 3 illustrates the process of $n(r)$ reconstruction in the case of high noise in the measurements of the extinction $L = L(\psi)$ shown in Fig. 4. Noise was modeled with the additive generation of uniformly distributed random numbers. Non-monotonic character of the variations in $L = L(\psi)$ gives rise to differentiation instability in determining the impact parameter. In order to avoid this instability we use here the polynomial approximation by the least squares method.

It should be noted that the profiles to be reconstructed can not be arbitrary. Thus, it follows from the theory of refraction that superrefraction must not contribute into the profile formation, i.e., the following condition should hold

$$[\mathrm{d}/(\mathrm{d}r)]\,[rn(r)] \geq 0.$$

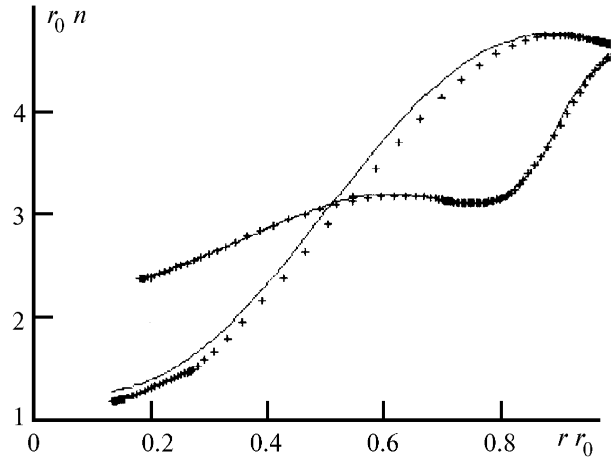

FIG. 2.

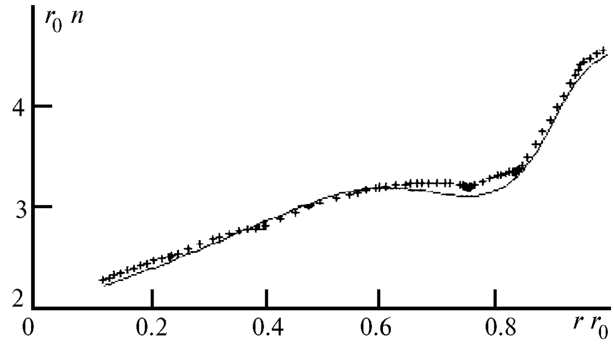

FIG. 3.

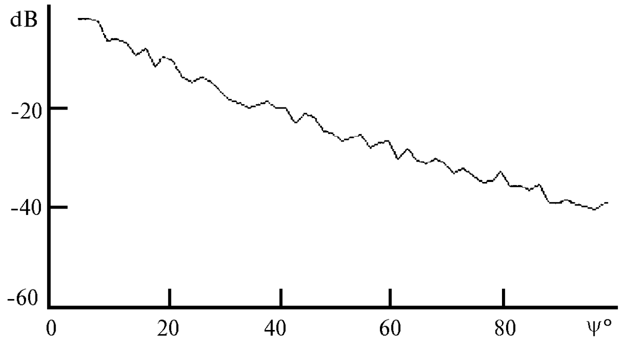

FIG. 4

Though directly applicable to AO, that may be interpreted as axially symmetric structures, the proposed method can be extended to arbitrary structures in the case of multiple view sensing.

5. CONCLUSION

The process of radiation propagation through strongly absorbing media can be studied in terms of amplitude trajectories giving minimum attenuation, whose description is analogous to that of refraction in the geometric optics. Assuming axial symmetry of the profile of linear absorption coefficient, and using inverse Abel transformation, this enables exact solution of the inverse problem on reconstructing the profile of absorption coefficient from the extinction measurements. The solution is stable in the presence of measurement noise.

The method can be used in remote diagnostics of the atmospheric pollutants, for constructing tomographic systems of sensing biological tissues, and other purposes. The sources may be optical, radio, or acoustic, and unnecessarily be coherent.

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