CHARACTERISTIC EQUATION FOR THE BACKSCATTERING PHASE MATRIX OF RECIPROCAL MEDIA

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The formal mathematical procedures have been established for finding the eigenvectors and eigenvalues of the single–point backscattering phase matrices (BPMs) of reciprocal media on the basis of analysis of symmetrical complex operators belonging to the congruent unitary transformation space.

It is well known[1,2] that the backscattering phase matrix (BPM) of reciprocal scatterers is always symmetrical and undergoes the congruent unitary transformation in going to a new polarization basis. Let BPM be specified by the four complex numbers:

$$S_g = \begin{pmatrix} \dot S_{11} & \dot S_{12} \\ \dot S_{21} & \dot S_{22} \end{pmatrix}, \tag{1}$$

in the Cartesian polarization basis, i.e., in the XY coordinate system, where $\dot S_{21} = \dot S_{12}$ (due to the symmetry). The basis vectors of the laboratory coordinate system are two linearly polarized cophased Jones vectors parallel to the X and Y axes, respectively. Then in the $(XY)'$ coordinate system and the previous Cartesian basis operator (1) takes the following form:

$$S_g^1 = \tilde R_\theta \, S_g \, R_\theta, \tag{2}$$

where $R_\theta = \begin{pmatrix} \cos\theta; & -\sin\theta \\ \sin\theta; & \cos\theta \end{pmatrix}$ is the rotation operator, θ is the angle between the $(XY)'$ coordinate system and the XY coordinate system in which the operator S_g (Eq. (2)) is represented, and the tilde denotes the transposition.

Operator (2) in the polarization basis, whose normal unit vectors $\mathbf{e}_1$ and $\mathbf{e}_2$ are the Jones vectors with major axes of the polarization ellipses being parallel to the X and Y axes of the $(XY)'$ coordinate system, respectively, with opposite directions of motion along identical ellipses of their polarization, has the form:

$$S_\varepsilon = (\widetilde{\mathbf{e}_1; \mathbf{e}_2}) \, S_g' \, (\mathbf{e}_1; \mathbf{e}_2) = F_\varepsilon \, S_g' \, F_\varepsilon, \tag{3}$$

where

$$\mathbf{e}_1 = \begin{pmatrix} \cos\varepsilon \\ j\sin\varepsilon \end{pmatrix}, \qquad \mathbf{e}_2 = \begin{pmatrix} j\sin\varepsilon \\ \cos\varepsilon \end{pmatrix}, \tag{4}$$

$$F_\varepsilon = \begin{pmatrix} \cos\varepsilon; & j\sin\varepsilon \\ j\sin\varepsilon; & \cos\varepsilon \end{pmatrix}. \tag{5}$$

The parameter ε in Eq. (5) specifies the absolute value of the ellipticity angle of the orthogonal basis vectors $\mathbf{e}_1$ and $\mathbf{e}_2$, with their orientation relative to the $(XY)'$ coordinate system remaining unchanged, and determines the ellipticity of the basis of the operator S_ε in Eq. (3). Substituting Eq. (2) into Eq. (3), we obtain

$$S_\varepsilon = \tilde F_\varepsilon \, \tilde R_\theta \, S_g \, R_\theta \, F_\varepsilon = \tilde L \, S_g \, L, \tag{6}$$

where $L = R_\theta F_\varepsilon$.

Thus, representation S_ε of the operator S_g in Eq. (1) in any basis with the ellipticity angle ε and the orientation angle θ (relative to the coordinate system in which the operator S_g is defined) has been given by general expression (6) in which the unitary operator L defines the rotation group in the three-dimensional space of the stereographic projection of the Jones vectors to the Poincare sphere, and the parameters ε and θ vividly and unambiguously pazametrize this group. It should be noted that this parametrization has the advantages over the Cayley-Klein parameters[3] that also parametrize unambiguously the indicated group by two independent parameters $\dot A$ and $\dot B$ in the form:

$$L = \begin{pmatrix} \dot A; & -\dot B^* \\ \dot B; & \dot A^* \end{pmatrix}, \tag{7}$$

where the complex numbers $\dot A$ and $\dot B$ satisfy the condition

$$\dot A \dot A^* + \dot B \dot B^* = 1, \tag{8}$$

and have no vivid physical meaning.

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Due to the symmetry of the operator S_g and the congruence of the transformation in Eq. (6), one can always find such parameters $\varepsilon = \varepsilon_0$ and $\theta = \theta_0$ for which the matrix S_ε takes the canonical (diagonal) form

$$S^0_\varepsilon = \begin{pmatrix} \dot\lambda_1 & 0 \\ 0 & \dot\lambda_2 \end{pmatrix} = \tilde F_{\varepsilon_0}\, \tilde R_{\theta_0}\, S_g\, R_{\theta_0}\, F_{\varepsilon_0} = \tilde L_0\, S_g\, L_0, \qquad (9)$$

where λ_1 and λ_2 are the eigenvalues of BPM, θ_0 and ε_0 are the eigenbasis parameters. Solving Eq. (9) for the operator S_g, we obtain

$$S_g = R_{\theta_0}\, F^*_{\varepsilon_0} \begin{pmatrix} \dot\lambda_1 & 0 \\ 0 & \dot\lambda_2 \end{pmatrix} F^+_{\varepsilon_0}\, \tilde R_{\theta_0} = L^*_0\, S^0_\varepsilon\, L^+_0, \qquad (10)$$

where the plus sign denotes Hermitian conjugation, and the asterisk denotes the complex conjugation. A vector signal observed at the exit from a field analyzer-shaper is related to a radiated signal by the expression:

$$\mathbf{U}_p = \tilde P\, S_g\, P\, \mathbf{U}_0, \qquad (11)$$

where P is the unitary operator of the analyzer–shaper determining its polarization properties (ellipticity and orientation of the measurement basis).[1]

Taking into account Eq. (10), expression (11) for the operator S_g takes the form:

$$\mathbf{U}_p = \tilde P\, L^*_0 \begin{pmatrix} \dot\lambda_1 & 0 \\ 0 & \dot\lambda_2 \end{pmatrix} L^+_0\, P\, \mathbf{U}_0\ , \qquad (12)$$

and for

$$P = L_0, \qquad (13)$$

which can be always satisfied by changing the ellipticity and orientation of the measurement basis (i.e., by changing the polarization properties of the analyzer), Eq. (12) takes the simplest form

$$\mathbf{U}_p = \begin{pmatrix} \dot\lambda_1 & 0 \\ 0 & \dot\lambda_2 \end{pmatrix} \mathbf{U}_0 = S^0_\varepsilon\, \mathbf{U}_0. \qquad (14)$$

Relation (14) means that the BPM eigenvectors corresponding to its eigenvalues $\dot\lambda_1$ and $\dot\lambda_2$ have the following form in the eigenbasis of the representation:

$$\mathbf{U}_{e1} = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \to \dot\lambda_1, \qquad \mathbf{U}_{e2} = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \to \dot\lambda_2. \qquad (15)$$

Obviously, the eigenvectors in the Cartesian basis of the description (in the basis of the operator S_g) in the XY coordinate system are reduced to the form

$$\mathbf{U}^g_{e1} = L_0 \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \qquad \mathbf{U}^g_{e2} = L_0 \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \qquad (16)$$

since the relation

$$S_g\, \mathbf{U}^g_{e1} = \dot\lambda_1 \left(\mathbf{U}^g_{e1}\right)^*, \qquad S_g\, \mathbf{U}^g_{e2} = \dot\lambda_2 \left(\mathbf{U}^g_{e2}\right)^* \qquad (17)$$

is valid for the operator S_g (see Eq. (10)) and vectors (16).

The vectors $\mathbf{U}^g_{e1}$ and $\mathbf{U}^g_{e2}$ are the unique vectors corresponding to the eigenvalues $\dot\lambda_1$ and $\dot\lambda_2$ of the operator S_g due to the unique operator L_0 inverse to the operator L^+_0 in Eq.(10) for S_g .

It is easy to make sure that the BPM eigenvectors $\mathbf{U}^g_{e1}$ and $\mathbf{U}^g_{e2}$ are always orthogonal

$$\left(\mathbf{U}^g_{e1}\right)^+ \mathbf{U}^g_{e2} = \left(\mathbf{U}^g_{e2}\right)^+ \mathbf{U}^g_{e1} = 0, \qquad (18)$$

since the relation

$$L^+_0\, L_0 = L_0\, L^+_0 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = I \qquad (19)$$

is valid.

Relation (17) contains the symbol of complex conjugation, which means that the BPM eigenvectors are contravariant to mapping of the XY coordinate system in which they are defined. This fact causes the difference of BPM from the transmission phase matrices (strictly forward scattering) belonging to the similarity transformation space. The relation defining their eigenvectors has the form

$$S\, \mathbf{e}_e = \lambda\, \mathbf{e}_e, \qquad (20)$$

which makes it possible to obtain the characteristic equation in the form

$$\mathrm{Det}\, \{S - \lambda\, I\} = 0. \qquad (21)$$

Starting from Eq. (17), the BPM eigenvectors have the form

$$S\, \mathbf{e}_e = \lambda\, \mathbf{e}^*_e. \qquad (22)$$

As follows from Eq. (16), in general the eigenvector $\mathbf{e}_e$ can be written in the form

$$\mathbf{e}_e = L_0 \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \qquad (23)$$

where the parameters θ_0 and ε_0 of the operator L_0 (see Eq.(6) for L) specify the parameters of the eigenbasis of BPM relative to the basis in which the operator S in Eq. (22) is represented. Substituting Eq. (23) into Eq. (22), we obtain

$$S\,L_0\begin{pmatrix}1\\0\end{pmatrix} = \lambda\,L_0^*\begin{pmatrix}1\\0\end{pmatrix}, \tag{24}$$

from which we can write the characteristic equation of BPM

$$\text{Det}\,\{S - \lambda\,L_0^*\,L_0^+\} = 0. \tag{25}$$

It follows from Eq. (6) for L and Eqs. (5) and (2) for F_ε and R_θ :

$$L_0 = R_{\theta_0}\,R_{\varepsilon_0} = \begin{pmatrix}\dot A; & -\dot B^*\\ \dot B; & \dot A^*\end{pmatrix}, \tag{26}$$

where

$$\dot A = \cos\varepsilon_0\,\cos\theta_0 - j\,\sin\varepsilon_0\,\sin\theta_0,$$
$$\dot B = \cos\varepsilon_0\,\sin\theta_0 + j\,\sin\varepsilon_0\,\cos\theta_0. \tag{27}$$

By substituting Eq. (26) into Eq. (25), after multiplication we obtain

$$\lambda^2 - \lambda\,[\text{Sp}\,S\,\cos2\varepsilon_0 - j\,\sin2\varepsilon_0\,\{(\dot S_{11} - \dot S_{22})\,\sin2\theta_0 -$$
$$- 2\,\dot S_{12}\,\cos2\theta_0\}] + \text{Det}\,S = 0, \tag{28}$$

where Sp and Det denote the spur and determinant of the operator S, respectively; $\dot S_{ij}$ are the elements of the operator S. Thus, relation (28) establishes the correspondence of the eigenvalues λ_1 and λ_2 with eigenbasis parameters ε_0 and θ_0 and elements $\dot S_{ij}$ of the matrix S in the form of two transcendental equations

$$\dot\lambda_1 = 0.5\,[\dot N + (\dot N^2 - 4\text{Det}\,S)^{1/2}],$$
$$\dot\lambda_2 = 0.5\,[\dot N - (\dot N^2 - 4\text{Det}\,S)^{1/2}], \tag{29}$$

where $\dot N = \text{Sp}\,S\cos2\varepsilon_0 - j\,\sin2\varepsilon_0\,\{(\dot S_{11} - \dot S_{22})\,\sin2\theta_0 - 2\dot S_{12}\,\cos2\theta_0\}$, and $\dot\lambda_1$ and $\dot\lambda_2$ (BPM eigenvalues) are solutions of square form (28) for λ.

A system of equations (29) contains six real unknowns: $|\dot\lambda_1|$, $|\dot\lambda_2|$, $\varphi_1 = \arg\dot\lambda_1$, $\varphi_2 = \arg\dot\lambda_2$, ε_0, and θ_0, and is equivalent to a system of four real equations that, at first glance, have no unambiguous solutions for six unknowns. Because the norm and the determinant of BPM are invariant for the basis parameters, a system of equations (29) can be completed by two relations

$$|\text{Det}\,S| = |\dot\lambda_1|\,|\dot\lambda_2|,$$
$$\|S\|^2 = |\dot\lambda_1|^2 + |\dot\lambda_2|^2. \tag{30}$$

It then follows that

$$|\dot\lambda_1| = \frac{1}{\sqrt2}\,[\|S\|^2 + (\|S\|^4 - 4|\,\text{Det}\,S|^2)^{1/2}]^{1/2}, \tag{31}$$

$$|\dot\lambda_2| = \frac{1}{\sqrt2}\,[\|S\|^2 - (\|S\|^4 - 4|\,\text{Det}\,S|^2)^{1/2}]^{1/2}, \tag{32}$$

where $\|S\| = \left(\sum_{\substack{i=1\\j=1}}^{2} |\dot S_{ij}|^2\right)^{1/2}$ is the Euclidean norm of BPM.

The system of equations (29) and (30) is unambiguously solvable for six unknowns $|\dot\lambda_1|$, $|\dot\lambda_2|$, φ_1, φ_2, ε_0, and θ_0. Generally speaking, relation (30) follows immediately from the system of equations (29), as their comprehensive analysis (which is beyond the scope of this paper) shows, and characteristic equation (28) provides complete information about the parameters λ_1, λ_2, ε_0, and θ_0 that parametrize sufficiently and nonredundantly BPM of a scattering object.

In summary the following conclusion can be drawn:

– BPM characteristic equation (28) unambiguously determines its eigenvalues and eigenbasis parameters (and hence its eigenvectors) and establishes formal mathematical rules for finding them as its solutions;

– the above–indicated formalism makes it possible to eliminate some discrepancies and evident errors encountered in some papers devoted to analysis of BPM (for example, see Ref. 4).

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