ON THE THEORY OF ASTRONOMIC REFRACTION IN A THREE–DIMENSIONALLY INHOMOGENEOUS ATMOSPHERE

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Expressions are developed within the three–dimensional model of the inhomogeneous atmosphere to calculate both the vertical and the lateral astronomic refraction. These describe refraction anomalies as functions of the zenith and azimuth angles, as well as the atmospheric parameters at the observation point.

Studies of refraction of electromagnetic waves in the atmosphere are highly important for increasing the accuracy of measuring linear angles in both the microwave and optical ranges.[1–3]

Many techniques have been suggested to estimate the refraction corrections. The largest group among them is that of techniques based on the approximate physical models of refraction, which yield comparatively simple mathematical expressions for computing such corrections. They use data on the problem geometry, on the parameters of the standard atmosphere, and on the values of meteorological parameters measured at the observation point.[1–5]

Another group of techniques consists of those based on the so–called reduction formulas. When developing them statistical techniques are employed to select empirical relations, which approximate the results of numerical simulations. In this approach the salient features of the physical models are not decisive; the principal criterion for the quality of reduction formulas is the agreement between the computational results yielded by those formulas and those from numerical simulations (see, e.g., Ref. 4).

An important place is also occupied by techniques of numerical retrieval of refraction corrections from data of radiosounding. These techniques make it possible to account, in most complete and concise manner, for the actual atmospheric parameter profiles along the sensing path. They are developed both for one–dimensional[5,6] and three–dimensional[7,8] models of the atmosphere.

Despite the long history of such studies and the numerous thechniques, certain problems of estimating optical refraction from angular observations have not yet found their adequate solution. In particular, some specific features in the anomalies of vertical refraction actually observed during experiments and depending on both the azimuth and zenith angles of the tracked target, are still discussed in publications, together with those of the particular behavior of the vertical, horizontal, and lateral refraction at large zenith angles, etc.

Reduction formulas cannot serve the purpose of analysis of such questions, since these are constructed for a one–dimensional atmospheric model. Numerical techniques do not fit this purpose either, the three–dimensional models of inhomogeneous profiles included,[7,8] because the final result of such calculations appears to be dependent on numerous parameters, so that the actual cause of this or that effect can hardly be isolated.

Within a wide range of azimuth and zenith angles the approach most feasible to study physical causes of the actual behavior of both the vertical and lateral refraction is the development of models containing only few parameters, which, on the one hand, account for the three–dimensional inhomogeneity of the atmosphere, and on the other, admit analytical expressions, illustrating explicitly the studied dependences. This approach is typical, for example, for studies described in Refs. 2 and 9.

In contrast to those studies which included some *a priori* set analytic profile of air refractivity (which is usually assumed exponential) we attempted to construct a theory, which would hold for an arbitrary three–dimensionally inhomogeneous–spatial distribution of air refractivity.

This theory is based on the integral description of the ray equations of geometric optics, formulated in Refs. 10 and 11. According to Ref. 11 the angles at which the beams arrive at final points of the trajectory are related (the effect of beam twisting being neglected) by the expression

$$n_L \mathbf{l}_L - n_0 \mathbf{l}_0 = \frac{\nabla n_0 + \nabla n_L}{2} L \, , \tag{1}$$

where n_0, n_L, ∇n_0, ∇n_L are the values of air refractivity and of its gradient, respectively, at the observation point and at the observed target; $\mathbf{l}_0$, $\mathbf{l}_L$ are the directions along which the beam arrives at these points (tangents to the trajectory); L is the trajectory length.

Limiting our treatment by astronomic refraction, i.e., assuming $n_L = 1$, $\nabla n_L = 0$, $\mathbf{l}_L = \mathbf{r}_L$ (where $\mathbf{r}_L$ is the true direction toward the observed target), we reduce the vector equation (1) to the form

$$\mathbf{r}_L - n_0 \mathbf{l}_0 = \nabla n_0 / 2L \, . \tag{2}$$

To simplify our calculations we introduce a Cartesian system of coordinates aiming its z axis to the zenith, its x axis to the south and its y axis to the east (their respective orths are $\mathbf{k}_z$, $\mathbf{k}_x$, $\mathbf{k}_y$) and present Eq. (2) in the form

$$\begin{aligned} &\mathbf{k}_z \cos z_\mathrm{t} + \mathbf{k}_x \sin z_\mathrm{t} \cos A_\mathrm{t} + \mathbf{k}_y \sin z_\mathrm{t} \sin A_\mathrm{t} - \\ &- n_0 \left[\mathbf{k}_z \cos z_\mathrm{a} + \mathbf{k}_x \sin z_\mathrm{a} \cos A_\mathrm{a} + \mathbf{k}_y \sin z_\mathrm{a} \sin A_\mathrm{a}\right] = \\ &= L/2 \left[\mathbf{k}_z \, g_\mathrm{v} + \mathbf{k}_x \, g_\mathrm{h} + \mathbf{k}_y \, g_\mathrm{l}\right] \, , \end{aligned} \tag{3}$$

where z_t and z_a are the true and apparent zenith angles; A_t and A_a are the true and apparent azimuths; g_v, g_h, and g_l are the vertical, horizontal, and lateral projections of the gradient of air refractivity at the observation point, respectively.

By multiplying Eq. (3) successively by $\mathbf{k}_z$, $\mathbf{k}_x$, and $\mathbf{k}_y$ (that is making a scalar operation) we obtain a system of three equations which, after the value L excluded from it, yields the following two equations:

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$$g_h(\cos z_t - n_0\cos z_a) = g_v(\sin z_t \cos A_t - n_0\sin z_a \cos A_a) ,$$

$$g_l(\cos z_t - n_0\cos z_a) = g_v(\sin z_t \sin A_t - n_0\sin z_a \sin A_a) . \quad (4)$$

Equations (4) can be interpreted as a system of equations for retrieving the angles of the vertical and lateral refraction (α and α_l, respectively), which are related to z_t, z_a and A_t, A_a as follows:

$$z_t = z_a + \alpha, \quad A_a = A_t + \alpha_l . \quad (5)$$

This retrieval is based on the use of the apparent zenith z_a and azimuth A_a angles, the refractive index n_0, and the projections of the refractivity gradient g_v, g_h, and g_l measured at the observation point.

From condition $z_t = z_a$ and $A_t = A_a$ and using the system of equations (4) we find the angles z_{a0} and A_{a0}, at which the refraction is absent, i.e., $\alpha = \alpha_l = 0$,

$$z_{a0} = \arctan\sqrt{g_h^2/g_v^2 + g_l^2/g_v^2} , \quad A_{a0} = \arctan(g_l/g_v) . \quad (6)$$

The absence of refraction in this case is explained by the coincidence of the direction along which the beam arrives with that of the gradient of refractivity, and as well known the beam curvature is equal to zero in this case.[12]

Thus, when there are horizontal gradients in the refractivity, zero vertical refraction take place at the angles z_{a0} and A_{a0} described by formula (6), and do not at $z_a = 0$. The numerical estimate of the angle z_{a0} can easily be obtained, e.g., for the values g_h, g_v related to the tilt of the layers with equal refractivity. According to Refs. 8 and 13 such tilts can reach 100″.

If one assumes that such a layer is tilted to the north, we have $g_h/g_v \simeq 100\cdot 5\cdot 10^{-6} \simeq 5\cdot 10^{-4}$ and $g_h/g_v \simeq 0$. Then $z_{a0} \simeq \arctan(5\cdot 10^{-4}) \simeq 5\cdot 10^{-4}$, i.e., the angle of zero vertical refraction can be displaced by 100″ from the direction to the zenith when such a horizontal gradient is present.

General solution of system (4) with the account for Eqs. (5) can be presented in the form

$$a = \arccos\left[n_0\cos z_a + \mathrm{T}\right] - z_a , \quad (7)$$

$$\alpha_l = A_a - \arctan\frac{\dfrac{g_l}{g_v}T + n_0\sin z_a \sin A_a}{\dfrac{g_h}{g_v}T + n_0\sin z_a \cos A_a} , \quad (8)$$

where

$$T = \frac{-n_0 S_A + \sqrt{n_0^2 S_A^2 - S_g(n_0^2 - 1)}}{S_g} , \quad (9)$$

$$S_A = \sin z_a\left(\frac{g_h}{g_v}\cos A_a + \frac{g_l}{g_v}\sin A_a\right) + \cos z_a , \quad (10)$$

$$S_g = 1 + \frac{g_h^2}{g_v^2} + \frac{g_l^2}{g_v^2} . \quad (11)$$

It is easy to check that in the limiting case in which $g_h = g_l = 0$ (a one–dimensional profile) the general solution of Eq. (7)–(10) is reduced to the form

$$\alpha_l = 0 , \ \alpha = \arccos\sqrt{n_0^2\cos^2 z_a - (n_0^2 - 1)} - z_a . \quad (12)$$

By expanding α in a power series $(n_0 - 1)$ and taking only the first term different from zero into account, one obtains the well–known expression from the Laplace–Oriani theorem[2,4]

$$\alpha = (n_0 - 1)\tan z_a . \quad (13)$$

It is evident from the formal mathematical point of view that formula (13) is valid, only for those angles at which the expression under the root sign in Eq. (12) is non–negative

$$n_0^2\cos^2 z_a - (n_0^2 - 1) \ge 0 , \quad (14)$$

from this condition we have

$$z_a \le \arccos\sqrt{\frac{n_0^2 - 1}{n_0^2}} . \quad (15)$$

Note that despite quite a large value of $z_a^{\lim}$ (the limit angle defined by the criterion (15) $z_a^{\lim} \simeq 88°$) theorem (13) practically guarantees a satisfactory accuracy only within $z_a < 70°$ (see Refs. 1–5).

However, even within this range, the Laplace––Oriani theorem fails to be adequate for accounting for the refraction effects. In particular, it does not describe the effect of the shift of the point of zero refraction, which is predicted by formulas (6).

Taking the horizontal components of refractivity gradients in the general solution of Eqs. (7)–(11) into account one can significantly improve the level of adequacy of the refraction model and broaden the range of angles $z \le z_a^{\lim}$. In the case of $z_a = 0$ and $A_a = 0$ the asymptotics of the general solution takes the form

$$\alpha \simeq -(n_0 - 1)\sqrt{g_h^2/g_v^2 + g_l^2/g_v^2} , \quad (16)$$

$$\alpha_l \simeq -\arctan(g_l/g_h) . \quad (17)$$

The negative values of refraction at zero observation angles result from the fact that according to Eq. (6) the point of zero refraction is shifted from the direction to the zenith determined by the angles z_{a0} and A_{a0}.

At moderate values of the angles z_a the value α can be expanded in a power series $(n_0 - 1)$ limited to the first term different from zero

$$\alpha \simeq (n_0 - 1)\frac{\sin z_a - \cos z_a\left[\dfrac{g_h}{g_v}\cos A_a + \dfrac{g_l}{g_v}\sin A_a\right]}{\cos z_a + \sin z_a\left[\dfrac{g_h}{g_v}\cos A_a + \dfrac{g_l}{g_v}\sin A_a\right]} . \quad (18)$$

It can be seen that in the absence of horizontal gradients ($g_h = g_S = 0$) expression (13) of the Laplace – Oriani theorem follows from relation (18).

Formula (18) makes it possible to derive a generalized expression covering the well–known descriptions of the refraction anomalies,[11,13] which follow from relation (18) at $g_l = 0$ (on subtracting the value $(n_0 - 1)\tan z_a$). In contrast to expressions derived in Refs. 11 and 13, formula (18), as well as general formulas (7)–(11), make it possible to present the anomaly of refraction as a function of the horizontal g_h and lateral g_l gradients of the refractivity, and the apparent azimuth A_a and zenith z_a angles. In particular, the expression for the horizontal α_a anomaly in refraction following from formula (18), has the form

$$\alpha_a \simeq -(n_0 - 1) \times \times \frac{\frac{g_h}{g_v}\cos A_a + \frac{g_l}{g_v}\sin A_a}{\cos^2 z_a \left[1 + \tan z_a \left(\frac{g_h}{g_v}\cos A_a + \frac{g_l}{g_v}\sin A_a\right)\right]}. \quad (19)$$

A similar relation can be obtained for the lateral refraction (the azimuth correction)

$$\alpha_l \simeq (n_0 - 1) \times \times \frac{\frac{g_h}{g_v}\sin A_a - \frac{g_l}{g_v}\cos A_a}{\sin z_a \cos z_a \left[1 + \tan z_a \left(\frac{g_h}{g_v}\cos A_a + \frac{g_l}{g_v}\sin A_a\right)\right]}. \quad (20)$$

Thus, formulas (19) and (20) generalize the well–known expressions[11,13] for refraction anomalies, expanding them to cover the case when the lateral gradient of the refraction is different from zero ($g_l \neq 0$). These formulas are applicable to the case of angles far from the limit $z_a \to 0$ and $z_a = \pi/2$. (One can see from Eq. (13) that as $z_a \to 0$ the value α_l should not depend on $(n_0 - 1)$ at all.) For the directions close to the horizon one should use general Eqs. (7) and (8).

The fact should also be kept in mind that the presence of square root in formula (9) imposes a limitation on the angles z_a in the vicinity of $z_a \simeq \pi/2$ (similar to the case of the Laplace–Oriani theorem). The condition according to which the expression under the root sign should remain non–negative

$$n_0^2 S_A^2 - S_g (n_0^2 - 1) \geq 0 \quad (21)$$

results in the following limiting value of the angle $z_a^{\lim}$, at which formulas (7)–(9) are yet valid:

$$z_a^{\lim} = \arccos \sqrt{\frac{n_0^2 - 1}{n_0^2} \frac{1 + \frac{g_h^2}{g_v^2} + \frac{g_l^2}{g_v^2}}{1 + \left(\frac{g_h}{g_v}\cos A_a + \frac{g_l}{g_v}\sin A_a\right)^2}} + \arccos \frac{1}{\sqrt{1 + \left(\frac{g_h}{g_v}\cos A_a + \frac{g_l}{g_v}\sin A_a\right)^2}}. \quad (22)$$

When $g_h = g_l = 0$ formula (22) agrees with inequality (15). The analysis shows that with growing g_h and g_l the range of angles at which formulas (7)–(11) are valid increases (it becomes wider than the applicability range for the Laplace–Oriani theorem).

However, more definite conclusions on the accuracy and applicability limits of expressions developed in the present study can be formulated only after comparing the calculational results with reliable experimental data and with computational results obtained using alternative technique (such as that in Refs. 7, 8, and 14), which accounts accurately for the three–dimensional inhomogeneity of the atmosphere. Such comparisons should made carefully as concerning the adequate account for the azimuth dependence of angles of refraction.

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