ANALYTICAL TECHNIQUE FOR DETERMINING VECTOR ORTHOGONAL POLYNOMIALS OF THE PHASE GRADIENT FOR ARBITRARY GEOMETRY OF A RECEIVING APERTURE

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This paper presents an analytical technique for reconstructing the coefficients of phase expansion in its gradient. The geometry of a receiving aperture and the expansion basis may be arbitrary.

The adaptive optics methods are used to increase the efficiency of laser systems operating under conditions of atmospheric distortions. Measurements of the wavefront distortions and their subsequent correction make it possible to decrease essentially jitter, scintillation, and spreading of optical beams and images. To describe the phase distortions of light fields on atmospheric paths, their expansion in a system of basis functions defined as Fourier series

$$S_N(x, y) = \sum_{k=1}^{N} a_k\, \Psi_k(x, y) \tag{1}$$

is commonly used, where the expansion functions are the orthonormalized polynomials whose form is determined by the geometry of a receiving aperture. In particular, they are Zernike polynomials for round aperture, $e^{i(xn+ym)}$ or Hermite–Chebyshev polynomials for square aperture, etc.

The vector of the coefficients $\mathbf{A}=(a_1, a_2, a_3, \dots, a_k)^{\mathrm{T}}$ of Eq. (1) is calculated from the condition of minimum of the functional

$$\min \rho(S, S_N) = \sqrt{\int_D (S - S_N)^2\, \mathrm{d}\rho}\ , \tag{2}$$

where D is the area of a receiving aperture. Minimization of functional (2) is equivalent to solving the matrix equation

$$\Phi\mathbf{A} = \mathbf{B}\ , \tag{3}$$

where

$$\mathbf{B} = \{<S, \Psi_1>, <S, \Psi_2>, \dots\}^{\mathrm{T}}\ , \tag{4}$$

Ψ_i is the expansion basis, $\Phi = \| <\Psi_i, \Psi_j> \|$ is the quadratic matrix, and $<f, g> = \int fg\, \mathrm{d}^2\rho$ is the scalar product.

Due to specific character of square–law detection, sensors of interference and Hartmann types are usually used as wavefront sensors in phase–conjugated adaptive optical systems. They measure wavefront tilts (gradients) in a discrete set of points. Therefore, we know the phase gradient

$$\nabla S(x, y) = \sum_{k=1}^{N} a_k\, \nabla\, \Psi_k(x, y)\ . \tag{5}$$

In this case, the following matrix equation[1–4] is applied:

$$F\mathbf{A} = \mathbf{C}\ , \tag{6}$$

where $F = \| <\nabla\Psi_i, \nabla\Psi_j> \|$ is the quadratic matrix, and $\mathbf{C} = \| <\nabla S, \nabla\Psi_j> \|$ is the row matrix. We note that the matrix F contains experimentally measured phase gradients, and the system may be ill–conditioned, i.e., its determinant may be close to zero. In this case, it is difficult to reconstruct the phase expansion coefficients. I offer an analytical approach to the determination of the phase expansion coefficients when the phase gradient is known.

First of all, let us find the relation between the phase and its gradient. The Borel–Pompey formula[5] is valid for the complex function $f(z) = S(x, y) + iV(x, y)$:

$$\frac{1}{2\pi i}\int_\Gamma \frac{f(z)\,\mathrm{d}z}{t-z} - \frac{1}{\pi}\int\!\!\int \frac{\partial f}{\partial z^*}\frac{\mathrm{d}\xi\,\mathrm{d}\eta}{t-z} = \begin{cases} 0\ , & z \in D_1, \\ f(z), & z \in D, \end{cases} \tag{7}$$

where $x, y \in \bar{D}$, $D_1 = R^2/\bar{D}$, and R^2 is a two–dimensional real Euclidean space.

Let D be the area of an aperture, and Γ be its boundary. Assuming that the phase is equal to zero at the boundary, we may rewrite Eq. (7) as

$$-\frac{1}{\pi}\int\!\!\int \frac{\partial f}{\partial z^*}\frac{\mathrm{d}\xi\,\mathrm{d}\eta}{t-z} = f(z)\ , \tag{8}$$

where $\partial/\partial z^* = (1/2)\,[\partial/\partial x] + i\,[\partial/\partial y]$. By virtue of linear independence between real and imaginary parts, we may rewrite Eq. (8) retaining only its real part:

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$$S(x, y) = -\frac{1}{\pi}\,\mathrm{Re}\int\int \frac{\partial f}{\partial z^*}\frac{\mathrm{d}\xi\,\mathrm{d}\eta}{t - z} =$$

$$= -\frac{1}{2\pi}\int\int_D \frac{\frac{\partial S}{\partial \xi}(x - \xi) + \frac{\partial S}{\partial \eta}(y - \eta)}{(x - \xi)^2 + (y - \eta)^2}\,\mathrm{d}\xi\;\mathrm{d}\eta \;=$$

$$= <\nabla S, 1/z>, \qquad (9)$$

where $1/z$ is the fundamental solution of the Cauchy–Riemann operator. Since the elements of the vector **A** are defined as

$$a_k = <\Psi_k, S>, \qquad (10)$$

substituting Eq. (9) into Eq. (10), we obtain

$$a_k = <\Psi_k, <\nabla S, 1/z>> = <\nabla S, <\Psi_k, 1/z>> = <\nabla S, \mathbf{G}_k>, \qquad (11)$$

where $\mathbf{G}_k$ are the vector polynomials defined by the relation:

$$\mathbf{G}_k = <\Psi_k, 1/z>, \qquad (12)$$

or in expanded form

$$\mathbf{G}_k = \{G_{kx}, G_{ky}\},$$

$$G_{kx}(x_0, y_0) = \int\int_D \frac{\Psi_k(x, y)(x - x_0)}{(x - x_0)^2 + (y - y_0)^2}\,\mathrm{d}x\,\mathrm{d}y, \qquad (13)$$

$$G_{ky}(x_0, y_0) = \int\int_D \frac{\Psi_k(x, y)(y - y_0)}{(x - x_0)^2 + (y - y_0)^2}\,\mathrm{d}x\,\mathrm{d}y. \qquad (14)$$

We note that all polynomials are orthogonal: $<\mathbf{G}_n, \mathbf{G}_m> = \delta_{nm}$, which makes the determination of the phase expansion coefficients much simpler.

Therefore, in order to determine the coefficients of phase expansion (1) on the basis Ψ_k in its gradient, it is necessary to expand the phase gradient ∇S into a series in vector polynomials $\mathbf{G}_k$ being the convolution of the basis functions with the Cauchy–Riemann fundamental solution. The domain of integration of the convolution should be determined by the aperture geometry.

The polynomials $\mathbf{G}_k$ obtained in the Cartesian coordinate system are presented in Refs. 6 and 7 for round aperture, when the phase was expanded in a system of the Zernike polynomials. The same polynomials were obtained in Ref. 8 for a particular case in the polar coordinate system by solving the boundary–value problem for round aperture.

Thus, the analytical technique for determining the phase expansion coefficients from phase gradients on arbitrary expansion basis for arbitrary aperture geometry has been proposed.

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