ENGINEER TECHNIQUE FOR CALCULATION OF INTENSITY DISTRIBUTION FROM THE TEMPERATURE OF A TARGET SURFACE

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Formulas for reconstructing the radiation intensity from the measured values of temperature of a target heated by the radiation are obtained. They are convenient for quick manual processing with calculators.

Reconstruction of the radiation intensity from the temperature of a heated surface (the problem of conversion of the boundary conditions) is the particular case of inverse problem of heat conduction.[1,2] This problem occurs when direct measurement of the radiation intensity, which heats the target, is unfeasible and only the temperature field of heated specimen is known.

In Refs. 1 and 2 the relationships for reconstruction of the intensity distribution from the surface temperature of a heated target are presented. Assuming the side surface to be heat–insulated and the intensity distribution over its front surface to be uniform, we can write the relation between the radiation intensity and a temperature on the target surface for following situations:

a) the temperature of the back surface of the target is maintained at its initial magnitude $T(t) = T_{\rm i}$

$$I(t) = \frac{\kappa}{L}\int\limits_0^t \frac{dT(\tau)}{d\tau}\left(1 + 2\sum_{n=1}^{\infty}\exp\left\{-\frac{a^2\,n^2\,\pi^2}{L^2}(t-\tau)\right\}\right)d\tau; \quad (1)$$

b) the back surface of the target is heat–insulated

$$I(t) = \frac{2\kappa}{L}\int\limits_0^t \frac{dT(\tau)}{d\tau}\sum_{n=1}^{\infty}\exp\left\{-\frac{a^2\,(2n-1)^2\,\pi^2}{4\,L^2}(t-\tau)\right\}d\tau\,, \quad (2)$$

where a^2 and k are the thermal diffusivity and the thermal conductivity coefficients, and L is the thickness of the target.

The algorithms for solving this problems is determined by design features of the temperature sensors and their positions at the target surface. In the simplest case we can measure the temperature over the target surface; in other cases the measurements of temperature are possible only in the depth of target at some distance from surface. Therefore, of interest is the two limiting cases, namely, the approximations of thick and thin targets, for which a thermophysical Fourier parameter $\mathrm{Fo} = a^2t/L^2$ takes the values

$$\mathrm{Fo} > 1 \quad (3)$$

and

$$\mathrm{Fo} < 1\,, \quad (4)$$

respectively.

Let us consider first the situation (a) when the temperature of the back surface of the target is maintained at its initial magnitude ($T(L,\,t) = T_{\rm i}$) under condition (3). We make use of the following heuristic considerations. The kernel of Eq. (1) represents the sum of deltoid peaks in the proximity of the upper integration limit. Assuming that $dT(t)/dt$ is the slowly variated function within the width of a peak we can factor it outside the integral sign without essential loss in accuracy. As a result, after integration we obtain

$$I(t) = \frac{\kappa}{L}\times$$
$$\times\left[T(t) - T_{\rm i} + \frac{2a^2L^2}{\pi^2}\frac{dT(\tau)}{d\tau}\sum_{n=1}^{\infty}\frac{1-\exp\left\{-\dfrac{a^2n^2\pi^2}{L^2}\,t\right\}}{n^2}\right]. \quad (5)$$

Neglecting the terms of higher–order infinitesimalicity and taking into account that

$$\sum_{n=1}^{\infty}\frac{1}{n^2} = \frac{\pi^2}{6}\,, \quad (6)$$

we obtain

$$I(t) = \frac{\kappa}{L}\left[T(t) - T_{\rm i} + \frac{1}{3}\frac{L^2}{a^2}\frac{d\,T(t)}{d\,t}\right]. \quad (7)$$

In a similar manner, taking into account the equality

$$\sum_{n=1}^{\infty}\frac{1}{(2\,n-1)^2} = \frac{\pi^2}{8}\,, \quad (8)$$

we obtain for the boundary condition (b)

$$I(t) = \frac{\kappa L}{a^2}\frac{d\,T(t)}{d\,t}\,. \quad (9)$$

Let consider the case of semi–infinite target (the inequality (4)). The direct substitution of the inequality (4) into Eqs. (1) and (2) makes the analysis of asymptotic very

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complicated. That is why let us make the preliminary rearrangements of the kernels of Eqs. (1) and (2).

The kernel of Eq. (1)

$$\theta(t) = 1 + 2 \sum_{n=1}^{\infty} \exp\left\{-\frac{a^2 n^2 \pi^2}{L^2} t\right\} \quad (10)$$

after simple rearrangements can be written as

$$\theta(t) = \int_0^t \sum_{n=-\infty}^{\infty} \exp\left\{-\frac{a^2 \pi^2 z}{L^2} t\right\} \delta(z - n)\, dz , \quad (11)$$

where $\delta(z)$ is the Dirac delta function.

In accordance with the theory of generalized functions we can establish the next identity[3,4]

$$\sum_{n=-\infty}^{\infty} \delta(z - n) = \sum_{n=-\infty}^{\infty} \exp(i\, n\, 2\pi\, z) . \quad (12)$$

Substituting the identity (12) into Eq. (11) and performing the Fourier transform we can write the following expression for the kernel:

$$\theta(t) = \frac{l}{a\sqrt{\pi\, t}} \left(1 + 2 \sum_{n=1}^{\infty} \exp\left\{-\frac{L^2 n^2}{a^2 t}\right\}\right) . \quad (13)$$

Taking into account the inequality (4) and keeping only the first term of expansion (13), after substituting it into Eq. (1) we obtain the well–known Abel's equation

$$I(t) = \frac{\kappa}{a\sqrt{\pi}} \int_0^t \frac{d\, T(\tau)}{d\, t} \frac{d\, t}{\sqrt{t - \tau}} . \quad (14)$$

The same equation also takes place for the boundary condition (b), but in the last case we should use in the intermediate rearrangements the following generalized identity[3,4]:

$$\sum_{n=-\infty}^{\infty} \delta\left(z - \frac{2n + 1}{2}\right) = 1 + 2 \sum_{n=1}^{\infty} (-1)^n \cos(n\, 2\pi\, t) . \quad (15)$$

Let represent the obtained limiting relationships in discrete form suitable for numerical calculation with a computer. Having divided the time interval $[0, t]$ into N sufficiently small intervals Δt, and denoted the discrete value of measured temperature on the surface by $T(m\Delta t) = T(t_m) = T_m$, $m = 1, 2, \ldots, N$, we can write the discrete analogs of limiting relationships at $\mathrm{Fo} > 1$ for conditions (a) and (b), respectively:

$$I(t_m) = \frac{\kappa}{L}\left[T(t_m) - T_{\mathrm{i}} + \frac{1}{3}\frac{L^2}{a^2}\frac{T(t_m) - T(t_{m-1})}{\Delta\, t}\right] , \quad (16)$$

$$I(t_m) = \frac{\kappa\, L\, (T(t_m) - T(t_{m-1}))}{a^2\, \Delta\, t} . \quad (17)$$

In semi–infinite approximation (at $\mathrm{Fo} < 1$) after approximated calculation of the integral in the right–hand part of Eq. (14), we obtain

$$\int_0^t \frac{d\, T(\tau)}{d\, \tau} \frac{d\, \tau}{\sqrt{t - \tau}} = \sum_{m=1}^{N} \int_{(m-1)\Delta t}^{m\Delta t} \frac{d\, T(\tau)}{d\, \tau} \frac{1}{\sqrt{t_N - \tau}}\, d\, \tau =$$
$$= \frac{2}{\sqrt{\Delta t}} \sum_{m=1}^{N} \frac{T_m - T_{m-1}}{\sqrt{N - m + 1} + \sqrt{N - m}} . \quad (18)$$

After rearrangements we obtain the discrete analog of Abel's equation (14)

$$I(N\, \Delta t) = \frac{\kappa}{a\sqrt{\pi}} \sum_{m=1}^{N} (T_m - T_{m-1})\, C_{N-m} , \quad (19)$$

where $C_j = 2(\sqrt{j + 1} - \sqrt{j})$. The area of applicability of the formula (19) is restricted by the condition $\mathrm{Fo} < 1$, but in practice we can use it when $\mathrm{Fo} \le 0.1$.

Thus, for the boundary conditions (a) and (b) in the approximations of thin and thick targets we obtain the simple analytical relationships and their discrete analogs which relate the radiation intensity with the temperature of heated target surface. This formulas are convenient for manual processing with calculators.

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