ESTIMATE OF THE POTENTIAL RESOLUTION OF PASSIVE METHODS OF IMAGE FORMATION THROUGH A TURBULENT ATMOSPHERE. IV. ADAPTIVE TELESCOPE WITH A HARTMANN SENSOR

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The limits on the parameters of an adaptive telescope with a Hartmann sensor and the conditions under which an extended exoatmospheric object can be observed and it is potentially possible to obtain a diffraction image are analyzed.

In our preceding papers[1–3] we analyzed one of the simplest, from the standpoint of practical implementation, approaches to the solution of the problem of viewing through the earth's atmosphere — speckle interferometry. This method, being based on successive recording and simultaneous statistical analysis of a series of short-exposure images, using large modern telescopes in the visible wavelength range, makes it possible to achieve a resolution almost at the diffraction limit. This, however, requires several tens to hundreds of thousands of initial images with a constant aspect angle of the object. This limits the range of priority application of speckle interferometry primarily to the problem of observing objects in high, including geostationary, orbits. At the same time, the problem of the formation of a diffraction image of a nonstationary object is a very important one. One effective method for solving this problem is to use adaptive telescopes. Appreciable progress has been made in the last few years in the development of the elemental base for such telescopes.[4]

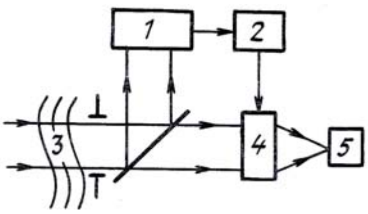

FIG. 1. Adaptive telescope with the Hartman sensor: 1) the Hartman sensor; 2) the computer; 3) the atmosphere; 4) the wavefront corrector; 5) the detector.

We shall estimate the potential efficiency of an adaptive optical system, based on Hartmann's method, for extended exoatmospheric objects (Figs. 1 and 2).[5] Hartmann's method (the phase distortions produced in the phase front by turbulence are determined from the displacements of images of the object at the foci of the subapertures covering uniformly the aperture of the telescope) is the most promising method for passive observation of such objects. Indeed, in Hartmann's method there are no fundamental limits on the size of the object and the width of the spectrum of the received radiation — such restrictions appear only as a result of the finite size of the region of isoplanatism and turbulence-induced dispersion effects in. the atmosphere.[6] This method has certain advantages in implementation over other methods.

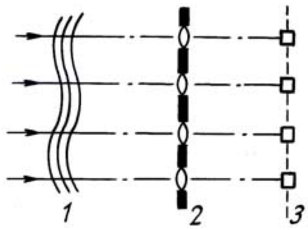

FIG. 2. The Hartman sensor: 1) the atmosphere; 2) the set of subapertures; 3) the set of coordinate detectors.

We emphasize that we shall be talking about the potential accuracy and its estimation, since only the most fundamental aspects will be taken into account and a number of assumptions will be made — Kolmogorov turbulence; circular aperture; hypothesis of "frozen-in" turbulence; layered atmosphere with respect to wind directions; finite number of subapertures covering the aperture with some density; quantum noise of detectors in the Hartmann sensor (other noises are ignored); and, finite image recording time of the sensor; errors in viewing the object as well as some mathematical assumptions, which, in particular, make it possible to leave unspecified the form of the subapertures, their arrangement, etc. The amplitude fluctuations of the field on the aperture, the background, the thermal noise of the detectors, the response of the apparatus implementing the algorithm reconstructing the phase front from its local tilts, the quality of the operation of different wavefront correctors, and possibly some other aspects, which could be of great but not fundamental importance, are ignored.

It should be noted that in this paper we shall employ the polynomial representation of the wavefront on the aperture, namely, an expansion in orthogonal circular Zernike polynomials $\varphi(r) = \sum_j C_j Z_j(r / R)$, where R is the radius of the aperture. This is one of

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many equivalent representations, but because the polynomials are close to the eigenfunctions of the Karhunen-Loéve integral equation whose kernel has the form of the correlation function of the phase, it makes it possible to relate approximately the finite number of subapertures, through the corresponding number of polynomials calculated for them, with the accuracy, determined by this number, of the determination of the wavefront.

The ultimate purpose of the adaptive optical system in this case is to obtain an image that is close to the diffraction image. As is well known, this is realized when the residual rms error in the phase is not worse than $\lambda/4$. This will be the starting point for the further analysis.

Based on the results of Refs. 7–9, we shall write down the following expression for the rms error in the estimate of the atmospheric phase, taking into account the finite number of subapertures, the errors in the measurements of the local tilts of the wavefront on the subapertures, and the errors in averaging with a finite measurement time:

$$\sigma^2_\varphi = \sigma^2_J + \sigma^2_\delta + \sigma^2_T. \quad (1)$$

This formula includes the following:

– the error in representing the wavefront on the aperture with a finite number J of Zernike polynomials

$$\sigma^2_J = \left[\sum_{j=J+1}^{\infty} C^0_j \right] \cdot \left(\frac{R}{r_0} \right)^{5/3}, \quad (2)$$

where C^0_j are the coefficients in the polynomial expansion;[7] r_0 is Fried's correlation radius of atmospheric distortions;

– the phase estimation error owing to the inaccuracy in the measurement of the tilts

$$\sigma^2_\delta = \sum_{j=1}^{J} \left[A^T A/\delta^2 + R_c^{-1} \right]^{-1}_{jj}, \quad (3)$$

A is a matrix, whose elements are the average, over the ith subaperture, tilts of the first surface, described by the jth Zernike polynomial;[8] δ is the accuracy with which the local tilts of the phase front on the subapertures are measured; and, R_c is the correlation matrix of the coefficients in the polynomial expansion; and,

– the error in representing the wavefront using polynomials whose coefficients are averaged over the time t (Ref. 9)

$$\sigma^2_T = \left[\sum_{j=1}^{J} C^T_j \right] \cdot \left(\frac{R}{r_0} \right)^{5/3} \cdot \frac{\langle v^2 \rangle T^2}{R^2}, \quad (4)$$

where C^T_j are constant coefficients, calculated in Ref. 9; $\langle v^2 \rangle$ is the squared velocity of turbulent nonuniformities averaged over the vertical distribution of the transverse wind velocity and atmospheric turbulence (the formula (4) was derived under the assumption that $\langle v \rangle^2 \ll \langle v^2 \rangle$ using the hypothesis that the turbulence is frozen-in, for polynomials whose radial part is of the order

$$n^2 \ll 13R^2\langle v^2 \rangle / T^2 \langle v^4 \rangle).$$

We shall now compare the terms (2)–(4) in the formula (1).

From the approximate expression given in Ref. 7 for the residual rms error of the phase with compensation of J aberrations it is easy to determine the required number of ideally reconstructed expansion coefficients, for which a given accuracy σ_j [rad] is obtained:

$$J \approx 0.9249\, \sigma_J^{-12/5} \cdot \left(\frac{R}{r_0} \right)^2, \quad J > 10. \quad (5)$$

It is obvious that the number of subapertures N must not be less than J. Thus the maximum size of a subaperture, corresponding to the minimum number of subapertures required, is virtually independent of the size of the aperture and is determined by Fried's parameter r_0. Suppose that for circular subapertures, filling the area of an aperture with coefficient β, $0 < \beta < 1$:

$$d_{max} \simeq 2R \sqrt{\frac{\beta}{N_{min}}} \simeq 2\, \sigma_J^{6/5} \beta^{1/2} r_0. \quad (6)$$

For example, for $\beta = 0.3$ and accuracy $\sigma_j = \lambda/10$ $d_{max} \simeq 0.64\ r_0$.

To estimate, with the help of the formula (3), the error in the reconstruction of the wavefront owing to the uncertainty in the measurement of the local tilts, we shall use the approximate expressions given in Ref. 8 for the elements of the matrix A and, in addition, using the sparseness and nearly diagonal form of the matrices A^TA and R_c, we shall include in Eq. (3) only their diagonal elements. Then

$$\sigma^2_\delta \simeq \sum_{j=1}^{J} \left[\left(\frac{\lambda}{2\pi R} \right)^2 \cdot \frac{N\alpha_j}{\delta^2} + \frac{1}{R_{c\,jj}} \right]^{-1} [\text{rad}^2], \quad (7)$$

where $\alpha_j = (n + 1)(n^2 + n - m^2)$, where n and m are the orders of the radial and angular parts of the jth polynomial. Here the first term takes into account the uncertainty of the measurement itself and the second term takes into account the magnitude of the aberrations themselves.

The error, owing is the quantum noise, in the determination of the tilt of the wavefront, matched with the image by the detector, within the subaperture is equal to[10]

$$\delta^2 = \frac{\rho^2}{8WS} [\text{rad}^2], \quad (8)$$

where ρ is the angular radius of the region of the image containing 90% of the energy of the received radiation, W is the average number of photoelectrons per unit area of the subaperture, and S is the area of the detector with four quandrants δ^2 is π times larger.) If the object is not resolved by a subaperture $\rho \simeq 2\lambda/d$; otherwise it can be assumed that $\pi\rho^2 \simeq \Omega$, where Ω is the solid angle covered by the object.

It can be shown that as j increases the second term in Eq. (7) increases more rapidly than the first term, i.e., the estimate of the higher order aberrations, compared with their rms magnitude, is less accurate. For the aberration of highest order (with index $j = J(n, m)$ – see the formula (5)) the ratio of the error in the estimation of the aberration to the rms value, depending on the order of the angular part m, is equal to

$$\left[\frac{2\pi R}{\lambda}\right]^2 \cdot \frac{\delta^2}{N_{\alpha_j}} \div R_{c_{JJ}} \simeq 0.6\, \sigma_J^{-16/9} \frac{\rho^2}{\lambda^2 \beta W} \times \left[\frac{R}{r_0}\right]^{-1} (m = 0) \ldots \ldots 0.4 \sigma_J^{-6/5} \frac{\rho^2}{\lambda^2 \beta W} \quad (m = n). \tag{9}$$

In the case when the ratios in Eq. (9) are small, σ_δ^2 is determined primarily by the errors in the measurement of the tilts and is limited by the quantity

$$\sigma_\delta^2 \le \left[\frac{2\pi R}{\lambda}\right]^2 \cdot \frac{\delta^2}{N} \cdot \sum_{j=1}^{J} \alpha_j^{-1} = \frac{\pi}{2} \cdot \frac{\rho^2}{\lambda^2 \beta W} \times \sum_{m,n} \frac{1}{(n+1)(n^2 + 2n - m^2)}, \tag{10}$$

in addition, Σ.... increases from 0.25 up to ≈ 1 as J changes from 1 to ∞. The conditions that the ratio (9) and the variance (10) be small are virtually equivalent, i.e., when the last aberration, taken into account based on Eq. (5), is estimated quite accurately compared with its rms value, the residual error owing to the uncertainty in the measurement of the wavefront tilts will be small compared with the wavelength.

As an example, we shall examine the standard situation: radiation from object, illuminated by the sun ($P = 1.6 \cdot 10^{26}$ photons/$m^3 \cdot s \cdot sr$), strikes the Hartmann system with a fill factor of 0.3 and a transmittance of 0.3, and is recorded at the wavelength 0.5 μm in a 0.03 μm band over a time of 30 ms matched with a detector with a quantum efficiency 0.2. If the number of aberrations determined ideally corresponds to $\sigma_j = 2\pi/10$, then to the ratio (9) of order 0.1 with $m = n$ (the relative error in the determination of aberrations of higher orders is small) there corresponds the ratio of the solid angle Ω, covered by the object, to the angular area $\pi\rho^2$ of the image of the object made by the subaperture equal to $\approx 2 \cdot 10^{-3}$. At the same time, the condition $\sigma_\delta \le 2\pi/10$. Thus for a subaperture 5 cm in diameter the minimum admissible value of Ω for the observed object is of the order of 0.1 (angular seconds)^2.

Rewriting the condition (10) in the form

$$\sigma_\delta \le \frac{2\pi}{\lambda} \left[\frac{S}{\pi\beta} \cdot \sum_{j=1}^{J} \alpha_j^{-1}\right]^{1/2} \cdot \delta, \tag{11}$$

it is not difficult to see that it is almost equal to the error, averaged over the subaperture, in the measurement of the local wavefront tilt; in addition, the factor $\sum_{j=1}^{J} \alpha_j^{-1}$ increases monotonically from 0.25 (only the tilt on one axis is taken into account) to ≈ 1 (all aberrations are taken into account) as the size of the aperture array increases. This result is in agreement with the results of a different approach, used in Refs. 11 and 12, where the wavefront is reconstructed from a collection of phase differences at neighboring sites of the array.

It is also interesting to find the accuracy with which the local tilts δ^* must be measured in order to obtain fixed values of σ_δ and σ_j. From the formulas (5) and (10) we find that for the minimum number of subapertures admissible for a given aperture and $J \gg 1$

$$\delta^* \simeq \frac{\lambda}{2\pi} \cdot \frac{\sigma_\delta}{\sigma_J^{6/5}} \cdot \frac{1}{r_0}. \tag{12}$$

Thus for $\sigma_\delta = \sigma_j = 2\pi/10$ we find $\delta^* \simeq \lambda/(6r_0)$. This result has a simple meaning. If a subaperture had the maximum possible diameter for achieving $\sigma_j = 2\pi/10$ (according to Eq. (6), $d_{max} \simeq 1.2r_0$ for $\beta = 1$), then the admissible error in determining the phase at the edges of the subaperture would be $\lambda/10$; then, in accordance with the relation (12), the value of σ_δ would also correspond to the accuracy $\lambda/10$. It Is easy to see that for $\beta = 0.3$, i.e., $d_{max} \simeq 0.64r_0$, $\delta^* = \lambda/6r_0$ is reached when the position of the center of the Hartmann image is measured with an accuracy of not less than 1/10 of its diffraction radius λ/d.

It is obvious that the value of δ (and σ_δ) will be affected by the error .in tracking the object with a telescope. In order that the motion of the object not result in an increase of σ_δ, the tracking error with respect to the angular velocity $\Delta\omega$ must satisfy the condition

$$\Delta\omega T \ll \delta, \tag{13}$$

where T is the time within which the Hartmann images are recorded. In the preceding case, for the minimum number of subapertures for a given aperture, $\sigma_\delta = \sigma_j = 2\pi/10$, $\lambda = 0.5$ μm, $r_0 = 5$ cm and $T = 3$ ms $\Delta\omega$ must be less than one angular minute/sec. From practice it is well known, however, that the tracking error with respect to the velocity with $\omega = 1 \ldots 2$ deg/s falls at the level of the characteristic vibrations of the drive and is equal to several angular seconds/sec.

Characteristically there is no fundamental need here for additional stabilization of the diffraction image formed by the adaptive telescope. Its position must be stabilized by the adaptive corrector itself.

Of course, there is a limit on the velocity of the object. The object should not leave the zone of isoplanatism while the Hartmann image is being recorded, the phase distribution is reconstructed, the correction is made, and the image is recorded. For example, if these operations are performed within 6 ms and the region of isoplanatism has a size of two angular seconds, the angular velocity of the object must not exceed 0.1 deg/s. If the velocity exceeds this value, the admissible duration of the series of operations listed above decreases proportionately, and this correspondingly increases the minimum angular area of the observed object. It should be noted that the maximum recording time can be increased by making an optimal forecast of the wavefront based on several preceding measurements.[13] The width of the Spectrum when the signal is recorded in the Hartmann sensor can also be increased. For example, increasing the width of the spectrum from 0.03 to 0.35 μm will make it possible to observe objects moving with velocities of up to 1.2 deg/s, but atmospheric dispersion limit the maximum zenith angle to 45 ... 60°.[6]

The last error is the time-averaging error σ_T (4). According to Ref. 9, for the conditions under which the formula (4) was derived,

$$C_j^T = \frac{11}{384}\left[n - \frac{5}{6}\right]\left[n + \frac{17}{6}\right] C_j^0 \frac{\langle v^2\rangle T^2}{R^2}\left[\frac{R}{r_0}\right]^{5/3}. \quad (14)$$

Then using the approximation of Ref. 5 for C_j^0 for large values of j ($j > 10$), we obtain

$$C_j^T \approx 4.6\cdot 10^{-2} j^{-5/6} \frac{\langle v^2\rangle T^2}{R^2}\left[\frac{R}{r_0}\right]^{5/3}. \quad (15)$$

Correspondingly, using the relation (5) we obtain

$$\sigma_T^2 \approx 0.342\, \sigma_J^{-0.31} \frac{\langle v^2\rangle T^2}{r_0^2}\ [\mathrm{rad}^2]. \quad (16)$$

Thus the averaging error is virtually independent of the size of the aperture array. It depends slightly on the fixed maximum accuracy, and it is determined by the ratio of the distance traversed by air nonuniformities moving with the average velocity to Fried's parameter r_0. Setting $\sigma_T = \sigma_j = 2\pi/10$, $\langle v^2\rangle = 64$ m^2/s^2, $r_0 = 5$ cm, we easily find that the maximum time $T \simeq 6$ ms.

We recall that the error σ_T, which we are studying here, is the rms error of the wavefront, approximated by a sum of polynomials and averaged over the observation time, from its temporal realization during this interval. In reality, however, over the time period determined in this manner (in other words, the time during which the atmosphere is "frozen"), not only is it necessary to construct the distribution of the wave front on the aperture, but the image obtained must also be corrected and recorded.

Analysis showed that in the employed model the size of the adaptive telescope is not restricted. This can be explained as follows. The phenomena studied above, which affect the efficiency of the adaptive telescope, occur over a characteristic distance r_0 (Fried's parameter), and the size of the subapertures is chosen to be proportional to this parameter. In this case the number of subapertures N increases as R^2. The error in the estimate of the phase distribution on an aperture based on N virtually independent measurements decreases as $1/\sqrt{N} \sim R^{-1}$ (this is seen especially clearly for the example of estimating the first aberration — the tilt of the wavefront, averaged over the aperture, based on N measured local slopes), i.e., with the same velocity as the diffraction limit of the aperture of the telescope λ/R.

In conclusion, we shall briefly repeat the results of the foregoing analysis and draw some conclusions.

1. The problem of obtaining close to diffraction-limited images of extended exoatmospheric objects is one of the problems in which adaptive optics could be useful. However there are some limitations, which are common to currently existing types of adaptive systems, on the maximum size and angular velocity of the object: the size of the zone of isoplanatism (several angular seconds) and the velocity corresponding to the traversal of the zone of isoplanatism over a time equal to the period of the adaptation loop. In any case, the adaptation loop cannot be significantly longer than the time during which the turbulence is "frozen-in," equal approximately to r_0/v_{wind} (of the order of several milliseconds).

2. The residual rms error $\lambda/10$, owing to the finite number N of independent measurements of the wavefront on the aperture, is obtained for $N \geq 2.8(R/r_0)^2$. In the Hartmann sensor this value of N corresponds to the number of subapertures, and the maximum diameter of the subapertures, chosen based on the condition of mechanical strength, is equal to $\simeq 0.6 \ldots 0.7\ r_0$.

3. Another component of the total error, determined by the finite accuracy of the measurements of the local tilts of the wavefront on the subapertures, also will not exceed $\lambda/10$, if this accuracy is not worse than $0.1/\sqrt{N}\ (\lambda/R)^2$ [rad]. Under typical conditions of observation the minimum angular size of the object is of the order of 0.1 (angular seconds).[2]

When these conditions are satisfied, the potential resolution of an adaptive telescope becomes comparable to that of a diffraction telescope.

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