SOLUTION OF THE RADIATION TRANSFER EQUATION BY THE METHOD OF SPHERICAL HARMONICS IN THE SMALL-ANGLE MODIFICATION

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A small-angle modification of the method of spherical harmonics for solving the radiation transfer equation is developed. The method is effective for calculating light fields in turbid media with anisotropic scattering. It is shown that the previously used forms of the small-angle approximation are a particular case of the approach developed in this paper.

Most real media, which are of practical interest from the standpoint of transmission of optical signals (sea water and atmospheric aerosol), are turbid media with strongly anisotropic scattering of radiation. The most efficient methods for calculating the light fields in such media are methods based on the solution of the radiation transfer equation (RTE) in the small-angle approximation.[1–3] This approach has been used to investigate the structure of the light fields of narrow beams,[4] nonstationary light fields,[5] optical-image transfer,[6] and other aspects of the theory of radiation transfer.

It has been found, however, that form of the small-angle approximation employed in Refs. 1–3, which is very effective for theoretical constructions, is not suitable for practical calculations, since the working expressions[1–6] are in the form of improper, very slowly converging integral of an oscillating function. A different approach to the solution of the RTE on the basis of the small-angle approximation is proposed in Refs. 7–9; this approach makes it possible to construct expressions that are convenient for practical calculations. This paper is devoted to the further development and extension of the ideas of Refs. 7–9.

We shall study the light field of an isotropic, monochromatic, point source of light (IMPS) in a turbid, radially nonuniform medium: the extinction coefficient is $\varepsilon = \varepsilon(R)$, the single-scattering albedo is $\Lambda = \Lambda(R)$, and the scattering phase function is $x = x(R; \hat{I}, \hat{l})$, where $\hat{R}$ is the radius vector from the center' of symmetry. (Here and below the symbol ' designates unit vectors. Within the small-angle approximation, the field of an IMPS, in accordance with the optical reciprocity theorem,[10] is equivalent to the distribution of spatial irradiance from a collimated point radiator in a stratified turbid medium. This fact is the basis for the investigation of systems for ranging and viewing in the atmosphere.

In this case the field is spherically symmetric: $L(\hat{R}, \hat{l}) = L(R, \mu)$, where $\mu = (\hat{l}, \hat{R}) / R$, and $\hat{l}$ is the direction in which the source is viewed. In the case of spherical symmetry, the RTE has the form

$$\mu \frac{\partial L}{\partial R} + \frac{1 - \mu^2}{R} \frac{\partial L}{\partial \mu} + \varepsilon(R) L(R, \mu) = \frac{\sigma(R)}{4\pi} \times$$
$$\times \oint L(R, \mu') x\left[R; \hat{l}, \hat{l}'\right] d\hat{l}', \tag{1}$$

where $\mu' = \left(\hat{l}', \hat{R}\right)$, $\sigma(r)\varepsilon(R)$.

The proposed method is based on expanding the solution $L(R, \mu)$ in Legendre polynomials $P_k(\mu)$, which are a complete set of orthogonal functions in the domain of the unknown function

$$L(R, \mu) = \sum_{k=0} \frac{2k + 1}{4\pi R^2} Y_k(R) P_k(\mu),$$

$$x\left[R; \hat{l}, \hat{l}'\right] = \sum_{k=0}^{\infty} \frac{2k + 1}{4\pi} x_k(R) P_k\left[\hat{l} \cdot \hat{l}'\right]. \tag{2}$$

this corresponds to the standard method of spherical harmonics (SH).

In this representation the RTE is converted into an infinite system of ordinary differential, equations:[10]

$$\frac{1}{2k + 1} \frac{d}{dR} \left[(k + 1) Y_{k+1} + k Y_{k-1}\right] + \frac{k(k + 1)}{2k + 1} \frac{1}{R}$$
$$\times \left(Y_{k+1} - Y_{k-1}\right) + b_k(R) Y_k(R) = 0,$$

where

$$b_k(R) = \varepsilon\left(R\right) - \sigma(R) \tilde{x}_k(R), \quad \tilde{x}_k(R) = x_k(R)/4\pi.$$

In the traditional method of SH the system (3) is truncated and solved under the condition $Y_k(R) = 0$, where k is greater than some number N (the so-called P_N approximation). However such a solution is extremely difficult to construct and can be constructed only for $N < 9$.[10] A general analysis of the properties of the solution of the RTE for the field of an IMPS in a uniform turbid medium shows[11] that the solution

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$L(R, \mu)$ has three singularities as a function of the angle at $\mu = 1$, namely, $\delta(1-\mu)$ in the zeroth-order term in the series expansion of $L(R, \mu)$ in terms of the order of scattering (the zeroth order scattering), $1\sqrt{1-\mu^2}$ for single scattering, and $-\ln(1-\mu)$ for double scattering. Owing to these properties of the solution the series (2) diverges at the point $\mu = 1$, and this makes it necessary to include a significant number of terms in the series in order to describe the solution in a neighborhood of the singular point (calculations based on the solution presented below showed that up to 3000 terms in the series were required). The orders of scattering above second order are smooth functions, and in order to describe the brightness body of the lowest orders the number of terms in the series expansion in the Legendre polynomials must be at least equal to the number of terms used in the expansion of the scattering phase function, which for real media is equal to several hundred. Therefore, since small optical distances $\tau = \int_0^R \varepsilon(r)dr$, where the bright body is formed primarily by small orders of scattering, are of interest from the practical standpoint, the traditional method of spherical harmonics, which strongly smoothes the singularity in the exact solution of the RTE, is of little use for practical calculations.

The main idea of the proposed modification of the method of SH is to determine the continuous dependence of the coefficients $Y_k(R)$ on the number k with the corresponding approximation of the values at the points $Y_{k-1}(R)$ and $Y_{k+1}(R)$; this makes it possible to reduce the system (3) to an analytically solvable equation of mathematical physics.

We shall introduce into the analysis the continuous function $Y(R, k)$, whose values at integer values of k correspond to $Y_k(R)$. Because $L(R, \mu)$ has singularities the function $Y(R, k)$ decreases slowly as k increases, so that $Y(R, k\pm1)$ can be expanded in a Taylor series in powers of k retaining two terms (linear approximation)

$$Y(R, k \pm 1) \simeq Y(R, k) \pm \frac{\partial Y}{\partial k}, \tag{4}$$

or the first three terms (parabolic approximation)

$$Y(R, k \pm 1) \simeq Y(R, k) \pm \frac{\partial Y}{\partial k} + \frac{1}{2}\frac{\partial^2 Y}{\partial k^2}. \tag{5}$$

Generally speaking, for series in Legendre polynomials, like for any functional series, the uncertainty relations and scaling theorems[12] are satisfied when for small values of the argument (angle) the terms in the series (2) with large numbers play a more important role. For this reason, the linear approximation (4) corresponds to $k \gg 1$ and is responsible for the description of singularities of the exact solution of the RTE (for example, as one can see from the definition (2), for a δ-function singularity $Y(R, k\pm1) = Y(R, k)$, while the parabolic approximation (5) also corresponds to small values of k, which are important when the sharpness of the bright body $L(R, \mu)$ is significantly reduced at large optical distances τ (quasideep light regime).

The representations (3) and (4) smooth the behavior of $Y_k(R)$ as a function of k. This requires a corresponding transformation of the boundary conditions also.[10] Since, in contrast to forward scattering, where $P_k(1) = 1$, in the case of backscattering $P_k(-1) = (-1)^k$, i.e., in a neighborhood of $\mu = 1$, the series (2) is sign alternating and is very sensitive to the behavior of the dependence of $Y_k(R)$ on k, neglecting backscattering for media with anisotropic scattering as a small parameter, we have the following boundary conditions:

$$\left.\begin{array}{l} Y_k(R) \longrightarrow 1 \text{ at } R \longrightarrow 0 \\ Y_k(R) \longrightarrow 0 \text{ at } R \longrightarrow \infty \end{array}\right\}, \quad k = 0, 1, 2, \ldots \tag{6}$$

We shall examine the linear approximation (4). In this case the system Eq. (3) assumes the following form when only the linear terms are retained:

$$\frac{\partial Y}{\partial R} + \frac{2k(k+1)}{2k+1}\frac{1}{R}\frac{\partial Y}{\partial k} + b(R, k)Y(R, k) = 0. \tag{7}$$

To solve Eq. (7) we make the substitution of variables

$$r = R, \ \xi = R/\sqrt{p}, \ p = k(k+1),$$

which results in the ordinary differential equation

$$\frac{\partial Y}{\partial r} + \left[\varepsilon(r) - \sigma(r)\tilde{x}(r, r/\xi)\right] Y(r, \xi) = 0,$$

which can be easily integrated. As a result we have

$$Y(R, k) = F\left[\frac{R}{\sqrt{p}}\right] \exp\left\{-\int_0^R \varepsilon(r)\,dr + \int_0^R \sigma(r)\,\tilde{x}\left[r, \frac{r}{R}\sqrt{p}\right] dr\right\}, \tag{8}$$

where $F(\xi)$ is an arbitrary smooth function.

To determine the terms in the series $Y_k(R)$ for small k, whose contribution increases as τ increases, we shall employ the parabolic approximation (5)

$$\frac{\partial Y}{\partial R} + \frac{1}{2k+1}\frac{\partial^2 Y}{\partial k \partial R} + \frac{1}{2}\frac{\partial^3 Y}{\partial k^2 \partial R} + \frac{2k(k+1)}{2k+1}\frac{1}{R}\frac{\partial Y}{\partial k} + b(R, k)Y(R, k) = 0. \tag{9}$$

The problem of solving Eq (9) for an arbitrary scattering phase function $\tilde{x}_k$ is no less complicated than the starting problem. In the case of strong anisotropy of scattering, however, for small k and α the representation

$$P_k(\cos\alpha) \simeq 1 - \frac{k(k+1)}{4}\cdot\alpha^2,$$

can be used. This gives the expression

$$\tilde{x}_k = \frac{1}{4\pi}\oint x\left[\hat{l}\hat{l}'\right]P_k\left[\hat{l},\ \hat{l}'\right]d\hat{l} \simeq 1 - \frac{k(k+1)}{4}\langle\alpha^2\rangle,$$

where $\langle\alpha^2\rangle = \frac{1}{2}\int\limits_0^\pi x(\alpha)\alpha^2\sin\alpha\, d\alpha$ is the average squared angle of single scattering. Correspondingly, we have for b

$$b(R,\ k) = \kappa(R) - k(k+1)D(R), \qquad (10)$$

where

$$\kappa(R) = \varepsilon(R) - \sigma(R), \quad D(R) = \sigma(R)\langle\alpha^2\rangle/4.$$

We shall seek the solution of Eq. (9) in the form

$$Y(R,\ k) = E_0(R)\exp\left[-\frac{k(k+1)}{4}S(R)\right], \qquad (11)$$

where E_0 and S are unknown functions of the radius vector R. It is easy to see that the function (11) smooths the singularity of the solution, and for this reason this approximation corresponds to large optical distances (quasideep light regime[13]).

Substituting Eq. (11) into Eq. (9) and using Eq. (10) we obtain an expression of the form

$$F_1(R) + F_2(R)k(k+1) + F_3(R)(k(k+1))^2 = 0$$

Since this expression holds for any k, this results in the system of equations $F_1(R) = 0$, $F_2(R) = 0$, $F_3(R) = 0$. The last expression is strictly valid for the deep regime, and analogously to Ref. 13 we shall neglect it in our calculation. Then under the condition of strong anisotropy of the bright body $S \ll 1$, retaining only infinitesimals of order S, we obtain

$$\begin{cases} \dfrac{1}{E_0}\dfrac{dE_0}{dR}\left[1 - \dfrac{S}{2}\right] - \dfrac{1}{2}\dfrac{dS}{dR} + \kappa(R) = 0, \\ \dfrac{dS}{dR}\left[1 - \dfrac{3}{2}S\right] - \dfrac{S^2}{2E_0}\dfrac{dE_0}{dR} - \dfrac{2S}{R} + 4D(R) = 0. \end{cases} \qquad (12)$$

The system of equations (12) cannot be solved analytically for arbitrary $k(R)$ and $D(R)$, so that we shall confine ourselves to the case of a uniform medium. For Eqs. (12) the boundary conditions (6) assume the following form in the limit $R \to 0$: $S(0) = 0$ and $E_0(0) = 1$. Then, based on the last two conditions and the condition $S \ll 1$, we obtain

$$S(R) = S_\infty \operatorname{cth}(CR) - \frac{2}{\kappa R}, \quad E_0(R) = \frac{CR}{\operatorname{sh}(CR)}e^{-\kappa R}, \qquad (13)$$

where $C = \sqrt{0.5\sigma<\alpha^2>\kappa}$, , $S_\infty = \sqrt{2\sigma<\alpha2>/\kappa}$.

We shall analyze the expressions obtained for the case of a uniform turbid medium. The solution (8) can be rewritten in the form

$$Y_k(R) = F\left[\frac{R}{\sqrt{k(k+1)}}\right]\exp\left[-\varepsilon R + \frac{\alpha R}{\sqrt{k(k+1)}}\times\right.$$
$$\left.\times\int\limits_0^{\sqrt{k(k+1)}} x(\zeta)d\zeta\right]. \qquad (14)$$

For small R the spatial illumination intensity

$$E_0(R) = \oint L\left[R,\ \hat{l}\right]d\hat{l} = \frac{Y_0(R)}{R^2}$$

decays in the same manner in both the linear (4) and parabolic (5) approximations: $Y_0 \sim \exp(-\kappa R)$. For large R the behavior of $Y_0(R)$ in the case of the linear approximation has the form $Y_0 \sim F(\xi)\cdot\exp(-\kappa R)$, and in the case of the parabolic approximation $-Y_0 \sim \exp[-(\kappa + C)R]$, i.e., it corresponds to extinction with the extinction coefficient at depth $\Gamma = \kappa + C$.

Comparing the results obtained with the well-known solutions of Refs. 1–3 and 13, we can assert that in the linear approximation the variance of the paths of the scattered photons is determined by $F(\xi)$, while the parabolic approximation corresponds to taking into account the variance of the paths using in Eq. (1) $\mu = \cos\alpha \simeq 1\ \alpha^2/2$. However, since $F(\xi)$ appears in the solution in the form of a constant cofactor, because it does not depend on r owing to the fact that in Eq. (7) the quadratic terms are dropped, the value of $F(\xi)$ cannot be determined from Eq. (7), and in what follows, based on the boundary conditions (6), we shall assume that $F(\xi) = 1$.

The properties of the solution (12)–(13) completely correspond to those of the solution given in Ref. 13, so that in what follows we shall not discuss it. To analyze the essential meaning of the approximation (7) and the relation between this approximation and other forms of the small-angle approximation, we shall study the case of a uniform layer confined between R_1 and R_2, $\Delta R = R - R_1$. Then from Eq. (8) we obtain

$$L(R,\ \mu) = \sum_{k=0}^{\infty}\frac{2k+1}{4\pi R^2}\exp\left\{-\varepsilon\Delta R + \right.$$
$$\left. + \frac{\sigma}{4\pi}\int\limits_{R_1}^{R} x\left[\frac{r}{R}\sqrt{k(k+1)}\right]dr\right\}P_k(\mu). \qquad (15)$$

The summation over k in Eq. (15) includes the term with $k = 0$, since for small angles ($\mu \to 1$) the contribution of the terms with small values of k is negligible.

We shall show that the previous forms of the small-angle approximation follow from Eq. (15) as particular cases:

1) The small-angle approximation with the form of the integral operator for the case of plane geometry remaining unchanged:[14,15] as $\Delta R/R \to 0$ we have from the theorem about the average

$$\int\limits_{R_1}^{R} x\left[\frac{r}{R}\sqrt{k(k+1)}\right]dr \simeq x\left[\left[1-\frac{\Delta R}{2R}\right]k\right]\Delta R \simeq x_k\Delta R,$$

whence

$$L(R,\ \mu) = \sum_{k=0}^{\infty}\frac{2k+1}{4\pi R^2}\exp\left\{-\varepsilon\Delta R\left[1-\Lambda\frac{x_k}{4\pi}\right]\right\}P_k(\mu).$$

2) The small-angle approximation with transformation of the integral term into a convolution integral:[1–3] restricting Eq. (15) to small angles, the following substitutions ($R_1 = 0$) are possible:

$$\sum_{k=0}^{\infty}\frac{2k+1}{4\pi} \longrightarrow \frac{1}{2\pi}\int\limits_0^{\infty} k\mathrm{d}k,$$

$$P_k(\cos\alpha) \simeq J_0(k\alpha), \quad \sqrt{k(k+1)} \simeq k,$$

$$L(R,\ \alpha) = \frac{e^{-\varepsilon R}}{2\pi R^2}\int\limits_0^{\infty}\exp\left\{\frac{\sigma}{4\pi}\int\limits_0^{R} x\left(\frac{r}{R}k\right)\mathrm{d}r\right\}J_0(k\alpha)k\mathrm{d}k,$$

where $J_0(k\alpha)$ is the zeroth-order Bessel function of the first kind.

For a strongly anisotropic scattering phase function

$$x_k = \frac{1}{2}\int\limits_{-1}^{1} x(\gamma)P_k(\cos\gamma)\ d\cos\gamma \simeq \frac{1}{2}\int\limits_0^{\pi} x(\gamma)J_0(k\gamma)\gamma\ d\gamma \simeq$$

$$\simeq \frac{1}{2}\int\limits_0^{\infty} x(\gamma)J_0(k\gamma)\gamma\ d\gamma.$$

Thus Eq. (15) is in complete agreement with Refs. 1–3.

3) Small-angle approximation with conversion of the integral term into a differential term — an equation of the Fokker-Planck type.[16]

For extremely anisotropic scattering or very small angles a representation analogous to the parabolic approximation is possible:

$$x_k \simeq 1 - \frac{k(k+1)}{4}\langle\alpha^2\rangle.$$

whence from Eq. (15) we obtain

$$L(R,\ \mu) = \sum_{k=0}^{\infty}\frac{2k+1}{4\pi R^2}\exp\left\{-\varepsilon(1-\Lambda)\Delta R - \frac{k(k+1)}{8}\Lambda\varepsilon\langle\alpha^2\rangle\Delta R\right\}P_k(\mu).$$

To analyze the properties of the solution obtained we shall expand it in orders of scattering; this is analogous to the expansion of Eq. (8) in a Taylor series in powers of σR. Correspondingly, for the first three orders of scattering we obtain

$$L_0(R,\ \mu) = \sum_{k=0}^{\infty}\frac{2k+1}{4\pi R^2}e^{-\varepsilon R}P_k(\mu) = \frac{e^{-\varepsilon R}}{2\pi R^2}\delta(1-\mu), \tag{16}$$

$$L_1(R,\ \mu) = \sum_{k=0}^{\infty}\frac{2k+1}{4\pi R^2}e^{-\varepsilon R}\times \left[\frac{\sigma R}{\sqrt{k(k+1)}}\int\limits_0^{\sqrt{k(k+1)}} x(\zeta)d\zeta\right]P_k(\mu), \tag{17}$$

$$L_2(R,\ \mu) = \sum_{k=0}^{\infty}\frac{2k+1}{4\pi R^2}e^{-\varepsilon R}\times \left[\frac{\sigma R}{\sqrt{k(k+1)}}\int\limits_0^{\sqrt{k(k+1)}} x(\zeta)d\zeta\right]^2 P_k(\mu). \tag{18}$$

As one can see from Eq. (16), the single-scattering solution contains a δ-function singularity. Since we are interested in the behavior of Eqs. (17) and (18) in the region of the singular point $\mu = 1$, terms of the series for which $k \gg 1$ will make the largest contribution. In accordance with Debye's principle of localization, in Mie's theory,[17] for an arbitrary scattering phase function, we can write

$$\int\limits_0^{\sqrt{k(k+1)}} x(\zeta)d\zeta \longrightarrow C_0 \quad \text{at } k \longrightarrow \infty,$$

where C_0 is a constant.

Therefore as $\mu \to 1$ the series (17) and (18) assume the form

$$L_1(R,\ \mu) = \frac{\sigma C_0}{2\pi R}e^{-\varepsilon R}\sum_{k=0}^{\infty}P_k(\mu),$$

$$L_2(R,\ \mu) = \frac{\sigma^2 C_0^2}{4\pi}e^{-\varepsilon R}\sum_{k=0}^{\infty}\left[\frac{1}{k+1}+\frac{1}{k}\right]P_k(\mu),$$

where the approximation $2\sqrt{k(k+1)} \simeq 2k+1$ for $k \gg 1$ was used.

Based on the properties of the generating function for Legendre polynomials, the last two series can be easily summed.

$$L_1(R,\ \mu) = \frac{\sigma C_0\ e^{-\varepsilon R}}{2\pi R\sqrt{2(1-\mu)}},$$

$$L_2(R,\ \mu) = \frac{\sigma^2 C_0^2}{4\pi} e^{-\varepsilon R}(-\ln(1 - \mu)).$$

Thus the solution Eq. (8) obtained above has all the properties of the exact solution. It is not difficult to show that the orders of scattering above second order are smooth functions.

From the foregoing analysis it is obvious that the above-described modification of the spherical harmonics (8) makes it possible to establish a single approach to the small-angle approximation. On the basis of such an approach the solution (8) is the most general form of the small-angle approximation and it neglects only the variance of the paths of the scattered photons and backscattering.

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