ASYMMETRY OF RADIATION ABSORPTION BY A HOMOGENEOUS SPHERE

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Analytical relations for the radiation power absorbed by the shadowed and illuminated hemispheres and for the asymmetry coefficient of absorption are derived within the framework of the Mie theory describing the scattering of a plane electromagnetic wave by a homogeneous sphere. The stable algorithm for calculating these quantities is constructed.

Absorption of electromagnetic radiation by aerosol is the classical problem in optics of disperse media. The Mie[1] theory is used to calculate the absorption coefficient in the simplest case of the spherical scatters. Here the particle is usually considered as a one entity and is described by the absorption cross section σ_{ab}. An alternative approach is the detailed description of the internal electromagnetic field distribution inside a sphere.[2] The first approach gives too little information about the absorption while the latter requires much computation time and often provides an information which is redundant and difficult for interpretation. In this connection an introduction of rather simple additional characteristics of absorption is of interest. For example, Refs. 3–5 are devoted to the calculation of σ_{ab} of the separate spherical sublayers inside the multilayered sphere. From our point of view, a convenient and information–bearing characteristic of this kind is the ratio η of the power W_s absorbed in the forward (shadowed) hemisphere to the corresponding value W_l for the backward (illuminated) hemisphere $\eta = W_s/W_l$. The calculation of W_{sl} by direct numerical integration of the function of the sources over the corresponding domains, especially for large particles, is hindered by complicated interference structure of the internal field. Therefore, more preferable way is to find analytical relations for the asymmetry parameter of the absorption η. This paper presents the solution of this problem.

Let us specify the geometry of the problem. A plane monochromatic ($e^{-i\omega t}$) linearly polarized electromagnetic wave with the amplitude E_0 (the vector **E** oscillates along the x axis) is incident in the positive direction of the z axis on a spherical homogeneous particle of radius R with the complex refractive index $m = N + i\kappa$ (the particle center coincides with the origin of the Cartesian coordinate system (x, y, z) and of the spherical coordinate system (r, θ, φ). Since the notations here will completely follow those used earlier in Ref. 1, first of all let us expand the internal electric field **E** and the magnetic field **H** in a system of vector spherical functions $\mathbf{M}^{(1)}_{\sigma 1n}$ and $\mathbf{N}^{(1)}_{\sigma 1n}$ (σ = e, 0 are the even and odd components) whose definition and properties can be found in Ref. 1:

$$\begin{pmatrix}\mathbf{E}\\ \sqrt{\mu_0/\varepsilon_0}\mathbf{H}\end{pmatrix} = E_0\begin{pmatrix}1\\ m\end{pmatrix}\sum_{n=1}^{\infty}\gamma_n \times$$

$$\times\left[\begin{pmatrix}c_n\\ -d_n\end{pmatrix}\mathbf{M}^{(1)}_{\substack{0\\e}1n}(mk_0\mathbf{r}) - i\begin{pmatrix}d_n\\ c_n\end{pmatrix}\mathbf{N}^{e}_{01n}(mk_0\mathbf{r})\right].$$

Here $\gamma_n = i^n/n(n + 1)$, $k_0 = 2\pi/\lambda$ is the wave number in the surrounding space, ε_0 and μ_0 are the electric and magnetic constants, **r** is the radius vector of the point inside the particle, c_n and d_n are the amplitude coefficients of the internal field (in contrast to Ref. 1, the factor $(2n + 1)$ enters into these coefficients). The components of the fields **E** and **H** in the spherical coordinate system have the form

$$\begin{pmatrix}E_r\\ \sqrt{\mu_0/\varepsilon_0}H_r\end{pmatrix} = \frac{E_0\sin\theta}{(m\rho a)^2}\begin{pmatrix}\cos\varphi\\ m\sin\varphi\end{pmatrix}\sum_{n=1}^{\infty} i^{n-1}\begin{pmatrix}Z_n\\ X_n\end{pmatrix}\pi_n\ ;$$

$$\begin{pmatrix}E_\theta\\ E_\varphi\end{pmatrix} = \frac{E_0}{m\rho a}\begin{pmatrix}\cos\varphi\\ -\sin\varphi\end{pmatrix}\sum_{n=1}^{\infty}\gamma_n\left[X_n\begin{pmatrix}\pi_n\\ \tau_n\end{pmatrix} - iV_n\begin{pmatrix}\tau_n\\ \pi_n\end{pmatrix}\right], \quad (1)$$

$$\begin{pmatrix}H_\theta\\ H_\varphi\end{pmatrix} = \frac{E_0}{\rho a}\sqrt{\frac{\varepsilon_0}{\mu_0}}\begin{pmatrix}\sin\varphi\\ \cos\varphi\end{pmatrix}\sum_{n=1}^{\infty}\gamma_n\left[Z_n\begin{pmatrix}\pi_n\\ \tau_n\end{pmatrix} - iY_n\begin{pmatrix}\tau_n\\ \pi_n\end{pmatrix}\right] \quad (2)$$

where the following notation has been introduced to simplify the derivations: $\rho = k_0R$ is the diffraction parameter, $a = r/R$ is the normalized radial distance, and π_n and τ_n are the angular functions of the argument $\mu = \cos\theta$.

$$\pi_n(\mu) = P^{(1)}_n(\mu)/\sqrt{1-\mu^2}\ ,\ \tau_n(\mu) = -\sqrt{1-\mu^2}\ dP^{(1)}_n(\mu)/d\mu$$

($P^{(1)}_n(\mu)$ is the associated Legendre polynomial)

$$X_n(a) = c_n\psi_n(m\rho a)\ ,\ Y_n(a) = c_n\psi'_n(m\rho a)\ ,$$

$$Z_n(a) = d_n\psi_n(m\rho a)\ ,\ V_n(a) = d_n\psi'_n(m\rho a)\ , \quad (3)$$

$\psi_n(m\rho a)$ is the Riccati–Bessel function, the prime denotes the derivative with respect to the argument.

In accordance with the Poynting theorem the power absorbed inside the volume surrounded by the closed surface S is equal to

$$W = -\frac{1}{2}\,\mathrm{Re}\int_S[\mathbf{EH}^*]\cdot\mathbf{n}dS\ ,$$

where **n** is the outward normal to the surface S, asterisk denotes the complex conjugation. It is obvious that $S_s = S_1 + S_3$ for the shadowed hemisphere and $S_l = S_2 + S_3$

for the illuminated hemisphere, where S_1 is the surface area of the shadowed hemisphere ($a = 1$, $0 \le \theta \le \pi/2$, and $0 \le \varphi \le 2\pi$), S_2 is the surface area of the illuminated hemisphere ($a = 1$, $\pi/2 \le \theta \le \pi$, and $0 \le \varphi \le 2\pi$), and S_3 is the area of the plane boundary between the shadowed and illuminated hemispheres ($0 \le a \le 1$, $\theta = \pi/2$, $0 \le \varphi \le 2\pi$). It is obvious also that the unit vector $\mathbf{e}_r$ is the outward normal to S_1 and S_2 while the vector $\pm\mathbf{e}_0$ (plus stands for S_s and minus is for S_l). If the corresponding integrals will be denoted as W_1, W_2, and W_3, the asymmetry parameter can be written in the form

$$\eta = (W_1 + W_3)/(W_2 - W_3)\ , \tag{4}$$

Let us proceed to the derivation of the relations for W_1, W_2, and W_3. In the expanded form the Poynting integral for W_1 has the form

$$W_1 = \frac{1}{2}\,\mathrm{Re}\int_0^{\pi/2}\int_0^{2\pi}\left(E_\varphi \mathbf{H}^*_\theta - E_\theta \mathbf{H}^*_\varphi\right)_{a=1} R^2 \sin\theta\, d\theta\, d\varphi\ .$$

Substituting expansions (1) and (2) and making the integration over the angle φ we obtain

$$W_1 = -A\,\mathrm{Re}\frac{1}{m}\int_0^1 \sum_{n=1}^{\infty}\sum_{l=1}^{\infty}\gamma_n\gamma_l^* \times$$

$$\times\left[(X_n\tau_n - iV_n\pi_n)(Z_l\pi_l - iY_l\tau_l)^* + \right.$$

$$\left. + (X_n\pi_n - iV_n\tau_n)(Z_l\tau_l - iY_l\pi_l)^*\right]_{a=1} d\mu\ ,$$

where $A = \pi E_0^2\sqrt{\varepsilon_0}/2\,k_0^2\sqrt{\mu_0}$. Combining the terms and changing the order of integration and summation we obtain the equation

$$W_1 = A\,\mathrm{Re}\frac{1}{m}\sum_{n=1}^{\infty}\sum_{l=1}^{\infty}\gamma_n\gamma_l^* \times$$

$$\times\left[i\beta_{nl}\int_0^1(\pi_n\pi_l + \tau_n\tau_l)d\mu - \alpha_{nl}\int_0^1(\pi_n\tau_l + \tau_n\pi_l)d\mu\right], \tag{5}$$

where

$$\alpha_{nl} = \left(X_n Z_l^* + V_n Y_l^*\right)_{a=1},\ \beta_{nl} = \left(V_n Z_l^* + X_n Y_l^*\right)_{a=1}\ . \tag{6}$$

It is well known[6] that

$$\int_0^1(\pi_n\tau_l + \tau_n\pi_l)d\mu = \pi_n(0)\pi_l(0)\ ; \tag{7}$$

$$\int_0^1(\pi_n\pi_l + \tau_n\tau_l)d\mu = \frac{n(n+1)\pi_n(0)\tau_l(0) - l(l+1)\pi_l(0)\tau_n(0)}{(n-l)(n+l+1)},\quad n \ne 1 \tag{8}$$

On the one hand, evaluating integral (8) for $n = l$ results in some difficulties, but using the procedure of integration by parts and equations for the Legendre polynomials gives the following relation:

$$\int_0^1(\pi_n^2 + \tau_n^2)d\mu = n^2(n+1)^2/(2n+1)\ . \tag{9}$$

On the other hand,

$$\pi_l(0) = \begin{cases}\nu_l(-1)^{(l-1)/2}, \\ 0,\end{cases}\quad \tau_l(0) = \begin{cases}0, & l \text{ is odd} \\ l\nu_{l+1}(-1)^{(-1)l/2}, & l \text{ is even}\end{cases} \tag{10}$$

where $\nu_l = l!!/(l-1)!!$. Thus, after some transformation instead of Eq. (5) we obtain

$$W_1 = A\,\mathrm{Re}\frac{1}{m}\left[i\sum_{n=1}^{\infty}\frac{\beta_{nn}}{2n+1} - \sum_{n=1}^{\infty}{}'\sum_{l=1}^{\infty}{}'\frac{\alpha_{nl}}{nl\nu_{n+1}\nu_{l+1}} + \right.$$

$$\left. + \sum_{n=2}^{\infty}{}''\sum_{l=1}^{\infty}{}'\frac{\nu_l(\beta_{ln} - \beta_{nl})}{\nu_n(n-l)(n+l+1)}\right], \tag{11}$$

where $\sum'$ and $\sum''$ denote the summation over the even and odd subscripts, respectively. Relation (11) for W_1 can be considered as final (it should be noted that when taking into account Eq. (6) the first double sum separates into the product of ordinary sums). The relation for W_2 can be derived from corresponding relation (5) for W_1 if the limits of integration are replaced by (–1, 0). But when such a substitution is made, integrals (7) and (8) change the sign while the sign of integral (9) remains unchanged. As a result, W_2 differs from W_1 (see Eq. (11)) only by the opposite signs before the double sums.

In the case of a plane boundary between the shadowed and illuminated hemispheres the Poynting integral has the form

$$W_3 = \frac{R^2}{2}\,\mathrm{Re}\int_0^1\int_0^{2\pi}(E_r\mathbf{H}^*_\varphi - E_\varphi\mathbf{H}^*_r)_{\mu=0}\, a\, da d\varphi\ . \tag{12}$$

Let us write out the components of the fields for $\mu = 0$ (taking into account Eq. (10))

$$\begin{pmatrix}E_r \\ \sqrt{\mu_0/\varepsilon_0}H_r\end{pmatrix}_{\mu=0} = \frac{E_0}{(m\rho a)^2}\begin{pmatrix}\cos\varphi \\ m\sin\varphi\end{pmatrix}\sum_{n=1}^{\infty}{}'\nu_n\begin{pmatrix}Z_n \\ X_n\end{pmatrix},$$

$$\begin{pmatrix}E_\varphi \\ \sqrt{\mu_0/\varepsilon_0}H_\varphi\end{pmatrix}_{\mu=0} = \frac{E_0}{m\rho a}\begin{pmatrix}-\sin\varphi \\ m\cos\varphi\end{pmatrix}\times$$

$$\times\left[\sum_{n=1}^{\infty}{}'\frac{1}{n\nu_{n+1}}\begin{pmatrix}V_n \\ Y_n\end{pmatrix} + \sum_{n=2}^{\infty}{}''\frac{1}{\nu_n}\begin{pmatrix}X_n \\ Z_n\end{pmatrix}\right]$$

and substitute them into Eq. (12). After integrating function (12) over the angle φ we obtain

$$W_3 = \frac{A}{\rho}\,\mathrm{Re}\int_0^1\frac{da}{a^2}\left[\frac{1}{m^2}\sum_{n=1}^{\infty}{}'\nu_n Z_n \times\right.$$

$$\times\left(\sum_{l=1}^{\infty}{}' Y_l\frac{\nu_l}{l(l+1)} + \sum_{l=2}^{\infty}{}''\frac{Z_l}{\nu_l}\right)^* +$$

$$+\frac{1}{|m|^2}\sum_{n=1}^{\infty}{}' \nu_n X_n\left(\sum_{l=1}^{\infty}{}' V_l \frac{\nu_l}{l(l+1)}+\sum_{l=2}^{\infty}{}'' \frac{X_l}{\nu_l}\right)^*\Bigg]. \qquad (13)$$

Unfortunately, we failed to eliminate the integral over a in formula (13), since the integrals of the form

$$\int_0^1 \psi_n(m\rho a)\,\psi_l^*(m\rho a)\frac{\mathrm{d}a}{a^2}\ (n \text{ is odd and } l \text{ is even})$$

have no closed analytical representation. We failed to obtain the recursive relations for n and l entering into Eq. (13).

Therefore, relation (13) for W_3 should be considered as final. Note that the domain of integration in Eq. (13) is rather "smooth" as far as the interference structure of the internal field is not virtually found here.[2]

Thus, relations (4), (11), and (13) make it possible to calculate σ_{ab} analytically for the shadowed and illuminated hemispheres and the asymmetry coefficient of absorption η for a homogeneous sphere. The results of calculation of integral (13) (or more correctly, the functions X_n, Y_n, Z_n, and V_n at the integrand points a_i) make it possible to calculate simultaneously the function $W(a)$ which describes the power absorbed by the spherical volume of the radius aR (Refs. 3 and 4)

$$W(a) = 2A\mathrm{Re}\left\{\frac{i}{m}\sum_{n=1}^{\infty}(2n+1)^{-1}\left[V_n(a)Z_n^* - X_n(a)Y_n^*\right]\right\}, \qquad (14)$$

as well as the normalized intensity of a local electric field $B(a)$ (Ref. 4) averaged over the angles θ and φ

$$B(a)=\frac{1}{4\pi}\int_0^{2\pi}\int_0^{\pi}B(a,\theta,\varphi)\sin\theta\mathrm{d}\theta\mathrm{d}\varphi = \frac{1}{2|m|^2\rho^2a^2}\sum_{n=1}^{\infty}(2n+1)^{-1}\times$$

$$\times\left[\frac{n(n+1)}{|m|^2\rho^2a^2}|Z_n(a)|^2+|V_n(a)|^2+|X_n(a)|^2\right]. \qquad (15)$$

Note that the equation derived in Ref. 4 is analogous to Eq. (15) but it has an error in the coefficient before the summation sign. The function $B(a)$ related to the function of the heat sources by the elementary formula $q(a) = 4\pi I_0 N\kappa B(a)/\lambda$, where I_0 is the intensity of the incident wave, can be used in solving the problem of heating of an aerosol particle upon exposure to an electromagnetic wave in a one–dimensional approximation especially for small values of I_0. For $a = 1$ relation (14) is virtually analogous to the Kattavar–Eisner formula for the absorption efficiency factor.

Let us discuss some calculating aspects of the problem. If we introduce the logarithmic derivatives of the Riccati–Bessel functions ψ_n and the Riccati–Hankel functions ξ_n

$$D_n(m\rho a)=\frac{\psi_n'(m\rho a)}{\psi_n(m\rho a)},\ D_n(m\rho)=\frac{\psi_n'(m\rho)}{\psi_n(m\rho)},\ G_n(\rho)=\frac{\nu_n'(\rho)}{\nu_n(\rho)},$$

and the ratio of the functions

$$R_n(m\rho a)=\frac{\psi_n(m\rho a)}{\psi_n(m\rho)},$$

and take into account the explicit relations for the amplitude coefficients c and d (Ref. 1), equations (3) become

$$X_n(a)=\frac{im(2n+1)R_n(m\rho a)}{\xi_n(\rho)[G_n(\rho)-mD_n(m\rho)]},\ Y_n(a)=X_n(a)D_n(m\rho a),$$

$$Z_n(a)=\frac{im(2n+1)R_n(m\rho a)}{\xi_n(\rho)[mG_n(\rho)-D_n(m\rho)]},\ V_n(a)=Z_n(a)D_n(m\rho a).$$

Thus, to calculate $W_{1,2}$ it is necessary to obtain a set of functions $D_n(m\rho)$, $G_n(\rho)$, $\xi_n(\rho)$, and ν_n while for numerical calculation of the integral W_3 we need the functions $D_n(m\rho a_i)$ and $R_n(m\rho a_i)$ at every point a_i of the interval of integration a = 0–1.

The functions $\xi_n(\rho)$, $G_n(\rho)$, and ν_n can be calculated by the forward recursion

$$\xi_{n+1}=\frac{2n+1}{\rho}\xi_n-\xi_{n-1},\ G_n=\frac{n}{\rho}+\frac{1}{n/\rho-G_{n-1}},\ \nu_{n+1}=\frac{n+1}{\nu_n}$$

with the initial values $\xi_0 = -ie^{i\rho}$, $\xi_1 = -(1+i/\rho)e^{i\rho}$ $G_0 = i$, and $\nu_1 = 1$ while the functions $D_n(m\rho)$ and $D_n(m\rho a_i)$ can be calculated by the backward recursion

$$D_{n-1}(z)=\frac{n}{z}-\frac{1}{n/z+D_n(z)}$$

starting from the fixed number $L = fL_W$, where L_W is the estimate of the number of the terms in the Mie series according to Ref. 8 and f is the empirical coefficient greater than unity. The appearance of the coefficient f is associated with the fact that the series in terms of the amplitude coefficients of the internal field converges slightly slower in comparison with the analogous series in terms of the coefficients of external field for which the estimate L_W was initially introduced. Our experience in calculations shows that $f\sim 1, 2$. The initial value D_L is calculated using the Lenz continued fractions.[9]

According to Ref. 10 the ratio of the functions $R_n(m\rho a_i)$ is calculated by the forward recursion.

$$R_n=\frac{D_n(m\rho)+n/m\rho}{n+m\rho a_i D_n(m\rho a_i)}\cdot m\rho a_i R_{n-1}.$$

It is convenient to represent the initial value R_0 in the form in which there are no factors of the form $\exp(\kappa\rho)$

$$R_0=\frac{B\sin(N\rho a_i)-iC\cos(N\rho a_i)}{U\sin(N\rho)-iQ\cos(N\rho_i)}\cdot e^{\kappa\rho(a_i-1)},$$

where

$$\left.\begin{matrix}B\\C\end{matrix}\right\}=e^{-2\kappa\rho a_i}\pm 1,\ \left.\begin{matrix}U\\Q\end{matrix}\right\}=e^{-2\kappa\rho}\pm 1.$$

which can easily overflow the computer.

Numerical integration of relation (13) was performed according to the Gauss quadrature formulas.[11] The number of nodes n of the quadrature was preliminary estimated based on our experience in calculating the internal fields and than refined based on the convergence of Eq. (13) when the number of integration points increases. To obtain four significant digits in W_3 n = 10–20 is sufficient in most cases being considered. Only sufficiently large and weakly absorbing particles ($\rho > 100$ and $\kappa < 10^{-4}$) are the exception. In this case we had to use $\overline{n}\sim 30$ and greater.

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REFERENCES

1. C.F. Bohren and D.R. Huffmen, *Absorption and Scattering of Light by Small Particles* (Interscience–Wirley, New York, 1986).
2. A.P. Prishivalko, *Internal Optical and Temperature Fields Inside the Light Scattering Particles* (Nauka i tekhnika, Minsk, 1983), 191 pp.
3. V. A. Babenko and L. G. Astaf'eva, Izvest. Akad. Nauk Belorussian SSR, Ser. Fiz. Mat. Nauk, No. 1, 55–60 (1985).
4. D.W. Mackowsky, R.A. Altenkirch, and M.P. Mengus, Appl. Optics **29**, No. 10, 1551–1559 (1990).
5. M. Sitarsky, Langmuir **3**, No. 1, 85–93 (1987).
6. G.D. Malushkov, Tr. Mosk. Inst. Radiotekhn. Elektron. Avtomat., No. 70, 153–157 (1974).
7. G. W. Kattawar and M. Eisner, Appl. Opt. **9**, No. 12, 2685–2690 (1970).
8. W.J. Wiscombe, Appl. Opt. **19**, No. 9, 1505–1509 (1980).
9. W.J. Lenz, Appl. Opt. **15**, No. 3, 668–671 (1976).
10. O.B. Toon and T.P. Ackerman, Appl. Opt. **20**, No. 20, 3657–3660 (1981).
11. M.A. Abramovits and I. Stigan, eds., *Reference Book on Special Functions* (Nauka, Moscow, 1979).