REPRESENTATION OF TRANSMISSION FUNCTIONS BY EXPONENTIAL SERIES

K.M. Firsov and A.B. Smirnov

Institute of Atmospheric Optics,
Siberian Branch of the Russian Academy of Sciences, Tomsk.
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Problem is solved of representing transmission functions by exponential series based on orthogonal exponential functions. Expansion coefficients are calculated from absorption coefficients by a direct technique. This model provides high accuracy calculation of transmission functions using only a few terms in the series.

One of concerns in climatological radiative transfer treatments is the development of economical parametrical model of transmission function in absorption bands of atmospheric gases. Of wide recent use, in this regard, has been the approximation of transmission functions by exponential series,[1–11] which seems to be attractive for a number of reasons[6]: 1) using this, the transmission becomes a multiplicative function of absorbing mass and thus can be computed more efficiently, 2) that substantially simplifies treatment of radiative transfer in a scattering medium,[12] and 3) the use of exponential series allows attainment of higher accuracy than when using special line models, since there is no need for specifying dependence of the transmission function upon the absorbing mass.

Among different methods of expanding the transmission functions in exponential series, the most interesting is that based on direct line–by–line (LBL) computations[3,4,10,11] and not requiring a preliminary model representation of the absorption spectra. This normally assumes the use of so–called K–function procedure which is well described elsewhere.[1–11] A thorough description is given in Ref. 11, in contrast to earlier, intuitive computational schemes like that in Ref. 10. Essentially, the algorithm[11] starts with the transmission function

$$\tau(U) = \frac{1}{\nu_2-\nu_1} \int\limits_{\nu_1}^{\nu_2} \exp\{-k(\nu)\,U\}\,d\nu , \tag{1}$$

where $k(\nu)$ is the absorption coefficient at frequency ν and U is the absorbing mass; which, through double Laplace transform, and by introducing new, scaled to 1, frequency g

$$g(\kappa) = \frac{1}{\nu_2-\nu_1} \int\limits_{\nu_1}^{\nu_2} W(\nu)\,d\nu ; \tag{2}$$

$$W(\nu) = \begin{cases} 1 , & k(\nu) < \mathrm{k} , \\ 0.5 , & k(\nu) = \mathrm{k} \\ 0 , & k(\nu) > \mathrm{k} . \end{cases} \tag{3}$$

is finally reduced to

$$\tau(U) = \int\limits_{0}^{1} \exp\{-\kappa(g)\,U\}\,dg , \tag{4}$$

where $\kappa(g)$ is the function inverse to (2). Because of transformation (1) through (4) the quickly oscillating function $k(\nu)$ converts into monotonic one, $\kappa(g)$ (see Figs. 1 and 2).

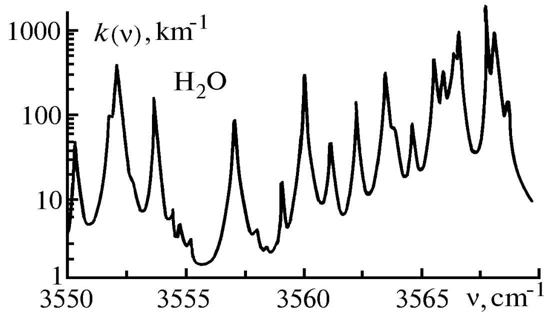

FIG. 1. H_2O *absorption spectrum calculated line–by–line using HITRAN–91 spectral line parameter values*[16] *for midlatitude summer atmosphere at* $H = 0$ km.

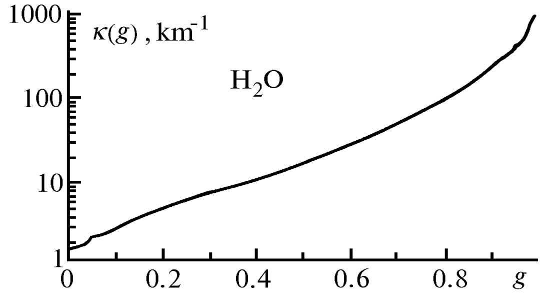

FIG. 2. H_2O *absorption coefficient versus normalized frequency;* $\kappa(g)$ *is converted from* $k(\nu)$, *as presented in Fig. 1.*

Next step toward obtaining the exponential series is to apply the appropriate quadrature formulas to the integral in formula (4), to arrive finally at the exponential series

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$$\tau(U) = \int_0^1 \exp\{-\kappa(g)\,U\}\,\mathrm{d}g = \sum_{i=1}^{n} a_i \exp\{-\kappa(g_i)\,U\}\,, \quad (5)$$

where a_i are expansion coefficients.

References 8 and 9 used Gaussian quadratures, in which case the nodes g_i represent the roots of orthogonal Legendre polynomials, while approximation (5) is highly accurate and becomes exact when its integrand is a polynomial of the order $2n - 1$. However the number of terms required in expansion (5) may sometimes be too large (more than twenty[8]), thus making this approach to obtaining exponential series inefficient. This is clearly so because the coefficients a_i and the nodes g_i in Eq. (5) do not depend on the form of the integrand. Also of note are companion Refs. 8 and 10 which use different integration of Eq. (4), but whose practical value is obscured by their empirical manner of choosing nodes and also by the fact that the fairly well developed methods of numerical integration are presently available.

The present paper focuses on the representation of transmission function by exponential series with the parameters determined from once calculated absorption coefficients $k(\nu)$ by minimization of the absolute rms error

$$\delta_n^2 = \int_0^\infty \left[\tau(U) - \sum_{k=1}^{n} a_k \exp\{-a_k U\}\right]^2 \mathrm{d}U\,. \quad (6)$$

Here, this problem is solved for homogeneous path; extension to inhomogeneous ones is readily made following Refs. 3 and 4.

As experience shows, approximation of a practically important functions is well accomplished by least squares fit using some functions of a narrower class, and much better than via interpolation polynomials.[13] That can be most easily done using orthogonal functions,[13] and precisely the class of exponential orthogonal functions are chosen here to represent transmission functions by exponential series, which, according to Refs. 1 and 14, have the form

$$D_m(U, \alpha_1, \alpha_2, ..., \alpha_m) = \sum_{k=1}^{m} C_{km} \exp\{-\alpha_k U\}\,, \quad (7)$$

where $D_m(U, \alpha_1, \alpha_2, ..., \alpha_m)$ is the mth orthogonal function, U is argument, $\alpha_1, \alpha_2, ..., \alpha_m$ are parameters, and the coefficients C_{km} are determined from orthogonality and normalization condition on functions D_m, namely

$$C_{km} = \frac{\sqrt{2\alpha_m}\,\prod_{i=1}^{m-1}(\alpha_k + \alpha_i)}{\prod_{i=1}^{k-1}(\alpha_k - \alpha_i)\prod_{j=k+1}^{m}(\alpha_k - \alpha_i)}\,. \quad (8)$$

Since the obtained system of functions is complete and the series uniformly converges,[14] it is possible to approximate $\tau(U)$ as

$$\tau(U) = \sum_{k=1}^{\infty} b_k\, D_k(U, \alpha_1, \alpha_2, ..., \alpha_k)\,; \quad (9)$$

$$b_k = \int_0^\infty \tau(U)\, D_k(U, \alpha_1, \alpha_2, ..., \alpha_k)\,\mathrm{d}U\,. \quad (10)$$

Upon collecting coefficients at equal exponents, we are led to series (5) with coefficients a_i now given as

$$a_i = \sum_{k=1}^{n} b_k\, C_{ik}\,. \quad (11)$$

Error from dropping terms after nth one is given by

$$\delta_n^2 = \int_0^\infty \tau^2(U)\,\mathrm{d}U - \sum_{k=1}^{n} b_k^2\,. \quad (12)$$

Thus, the problem of constructing series (5) is solved via minimization of Eq. (12) with respect to unknown coefficients α_k.

Since the integrands in Eqs. (1) and (4) have exponential forms, the coefficient b_k can be represented as

$$b_k = \sum_{i=1}^{k} C_{ik}\, J(\alpha_i)\,; \quad (13)$$

$$J(\alpha_i) = \frac{1}{\nu_2 - \nu_1}\int_{\nu_1}^{\nu_2} \frac{\mathrm{d}\nu}{k(\nu) + \alpha_i} = \int_0^1 \frac{\mathrm{d}g}{\mathrm{k}(g) + \alpha_i}\,; \quad (14)$$

while integral in Eq. (12) is also written in terms of absorption coefficient as

$$\int_0^\infty \tau^2(U)\,\mathrm{d}U = \frac{1}{(\nu_2 - \nu_1)^2}\int_{\nu_1}^{\nu_2}\mathrm{d}\nu \int_{\nu_1}^{\nu_2}\frac{\mathrm{d}\nu'}{k(\nu) + k(\nu')} =$$

$$= \int_0^1 \mathrm{d}g \int_0^1 \frac{\mathrm{d}g'}{\mathrm{k}(g) + \mathrm{k}(g')}\,. \quad (15)$$

Thus, the problem of obtaining exponential series in fact reduces to that of minimizing relation (12) with the help of variable coefficients α_i. As seen from Eqs. (14) and (15), this problem can in principle be solved without invoking k–representation, by evaluating the integrals from absorption coefficients using direct LBL technique. However, for efficient numerical implementation of the algorithm, the use of $\kappa(g)$ is recommended because it is a smooth function. Equations (14) and (15) were evaluated by means of quadrature Gaussian formulas using 25 points, that ensured no worse than 1% computation error for a range of transmission of 0.001 to 1.

Minimization of Eq. (12) was accomplished by an iteration procedure, specially devised and proved to be sufficiently fast and efficient for this purpose. Specifically, a zero approximation for first two coefficients is determined following the recommendations from Ref. 14; next, the contribution of the third term is included, and so forth.

The method was run at the 2.7 μm water vapor band and 15 μm carbon dioxide band. Absorption coefficients were computed with a fast LBL technique[15] using published spectral parameters.[16] Figures 3 and 4 show absolute and relative errors of representing H_2O transmission function by a three–, four–, and five–term expansion, each derived by a comparison with the direct computation; the results for CO_2 are analogous. From the figures it can be concluded that:

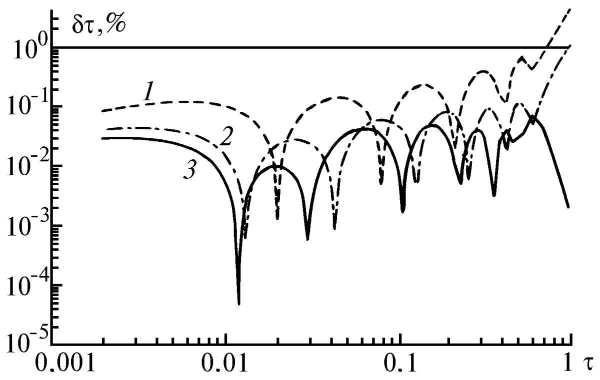

FIG. 3. Absolute error of representation of transmission function by exponential series, derived from a comparison to LBL results, for 3550– 3570 cm^{-1} *spectral band of* H_2O *and horizontal path in the midlatitude summer atmosphere at* H = 0 km. *Number of terms in the series* n = 3, 4, 5 *(curves 1, 2, 3).*

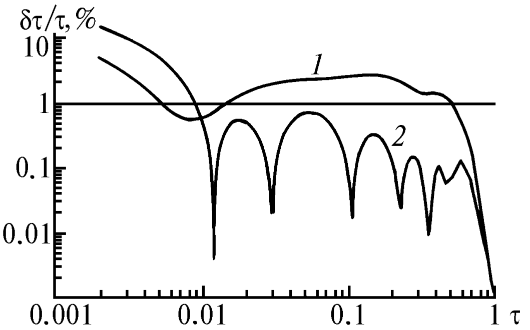

Fig. 4. Relative error from transmission function computation using Gaussian quadratures (number of nodes n = 5, *curve 1), and from the method adopted in this paper (number of terms in the series* n = 5, *curve 2), for same conditions as specified in Fig. 3.*

1) The errors are the less, the more expansion terms are included.

2) In a weakly absorptive case ($U \to 0$), the use of five–term expansion gives an acceptable accuracy, and much better than using three– or four–term one.

3) For the range of transmission function of 0.01 to 1, use of five–term expansion will give no more than 1% relative error, an acceptable precision in remote sensing applications.

4) The increased relative error for $\tau < 0.01$ stems from case of the specific orthogonal functions used in the study and, accordingly, from the form of functional (12). That can be overcome by either using more terms in the series, or employing other orthogonal functions with appropriate weights. In many applications, however, just working within transmission function range 0.01 to 1 will suffice.

5) The traditionally used technique for constructing exponential series[8,9] with the help of Gaussian quadratures is much more in error than that just outlined.

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REFERENCES

1. A.A. Dmitriev and G.M. Abakumova, Izv. Akad. Nauk SSSR, Ser. Fiz. Atmos. Okeana **4**, No. 1, 102–103 (1970).
2. L.D. Krasnokutskaya and T.A. Sushkevich, Izv. Akad. Nauk SSSR, Ser. Fiz. Atmos. Okeana **13**, No. 5, 505–515 (1977).
3. M.–D. Chou and A. Arking, J. Atm. Sci. **37**, No. 4, 855–867 (1980).
4. M.–D. Chou and A. Arking, J. Atm. Sci. **38**, No. 4, 798–807 (1981).
5. G.L. Stephens., Month. Weather Rev. **112**, 826–857 (1984).
6. Y. Fouquart, *Radiative Transfer in Climate Models. Physically Based Modeling and Simulation of Climate and Climatic Change*, ed. by M.E. Shlesinger (Academic Publishers, Kluwer, 1988), Part 1, pp. 223–283.
7. W.S. Wang and G.Yu. Shi, J. Quant. Spectrosc. Radiat. Transfer **39**, No. 5, 387–397 (1988).
8. R. Goody, R. West, L. Chen, and D. Crisp, J. Quant. Spectrosc. Radiat. Transfer **42**, No. 6, 539–550 (1989).
9. Ph. Riviere, A. Soufani, and J. Taine, J. Quant. Spectrosc. Radiat. Transfer **48**, No. 2, 187–203 (1992).
10. A.A. Lacis and V.A. Oinas, J. Geoph. Res. **96**, No. D5, 9027–9063 (1991).
11. S.D. Tvorogov, Atmos. Oceanic Opt. **7**, No. 3, 165–171 (1994).
12. E.M. Feigelson, *Radiant Heat Exchange and Clouds* (Gidrometeoizdat, Leningrad, 1970), 157 pp.
13. I.S. Berezin and N.P. Zhidkov, *Calculation Techniques* (Nauka, Moscow, 1959), Vol. 1, 421 pp.
14. A.A. Dmitriev, *Orthogonal Exponential Functions in Hydrometeorology* (Gidrometeoizdat, Leningrad, 1973), 120 pp.
15. A.A. Mitsel' and K.M. Firsov, Atmos. Oceanic Opt. **7**, No. 10, 779–780 (1994).
16. L.S. Rothman, R.R. Gamache, R.H. Tipping, et al., J. Quant. Spectrosc. Radiat. Transfer **48**, 469–507 (1992).