CALCULATION OF NONLINEAR ABERRATION DURING THERMAL BLOOMING OF WAVE BEAMS

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Based on the transfer equation for the intensity and the parabolic equation for the eikonal, a system of ordinary differential equations is obtained which describe the stationary propagation of axially symmetric light beams in media with an arbitrary nonlinearity under conditions of aberrational distortion of any prescribed order. Using the found recursion formulas relating the coefficients of series expansions over basic elementary functions of infinite series, the obtained system of equations is adapted for describing the wave aberrations up to the sixth order that occur during the propagation of Gaussian beams through media with a cubic nonlinearity and under conditions of thermal blooming. An analysis is carried out of the integrals of motion of the medium and of the differential equations themselves.

An analysis of numerical solutions of the parabolic equation describing light propagation in nonlinear media indicates the necessity of taking into account, besides the nonaberrational distortions, additional nonlinear aberrational distortions of spatially limited beams.[1,2] Nonlinear aberrations show up not only in near-focal regions, where their role is decisive, but because of the cumulative nature of the aberrations the behavior of the beam can differ strongly from the predictions of aberration theory already after the first nonlinear length.[3]

Investigations of nonlinear aberrations based on the parabolic partial differential equation represent a mathematically complicated problem and, hence, require the use of numerical methods. At the same time, numerical methods lack the advantages of an analytical description, which gives the most complete description of the phenomenon. The first attempt to carry out an aberrational analysis was undertaken in Ref. 4. However, many problems concerning both general aberrational methodology and the possibility of a detailed investigation of aberrational distortions of beams in media with an arbitrary nonlinearity mechanism remain open. In particular, the structure of the transverse beam profile, which was first elucidated in Ref. 5 on the basis of the approximation of spherical aberrations, remains unexplored.

In the present paper an aberrational theory of thermal blooming of arbitrary axially symmetric wave beams is presented, with the nonlinearity mechanism of the medium in general not specified. The theory generalizes the nonaberrational method and enables one, after passing over to a system of ordinary differential equations, to substantially simplify the use of numerical solution methods and make them effective. An aberrational analysis constructed on the basis of this theory has made it possible to investigate in detail many propagation features which escape the nonaberrational approach, such as limitation of the transverse size of Gaussian-1ike beams, the occurrence of a complicated annular structure of the beam cross section, the formation of an aberrational ring in a defocusing medium,[5] and so on.

Let us suppose that beam trajectories $\rho(z)$ are known for an axially symmetric beam as functions of .the longitudinal coordinate z. Then an addition to the eikonal of a plane wave $s(r)$ on the circles $r = (\rho, z)$ may be calculated using the relation $\frac{\partial s}{\partial \rho} = \frac{d\rho}{dz}$. For a beam with a parabolic wavefront we introduce the variable $\xi(r) = \frac{\rho}{a_0 f(z)}$, where a_0 is the initial characteristic beam radius and $f(z)$ is its current dimensionless radius without aberrations. Since $\frac{\partial s}{\partial \xi} = a_0 f\left(a_0 f'\xi + a_0 f \frac{d\xi}{dz}\right)$, integration gives

$$s(r) = s_0(z) + \rho^2 f'/2f + a_0^2 f^2 \int_0^{\xi} (d\xi/dz)d\xi \ , \tag{1}$$

where $s_0(z)$ is the additional phase shift due to the change in the wave propagation velocity.

In accounting for the aberrations the variable $\xi(r)$ is determined by a set of N dimensionless aberrational functions $A_n(z) = \{A_p(z)\}$, $1 \le p \le N$ $(N = 1, 2, \ldots)$. Let the derivative A' characterize (provided that f and all of the A_q, $1 \le q \le N$, $q \ne p$, are constant) the contribution of the distortions to the beam wavefront that are described by the power of the transverse coordinate $\rho^{2(p+1)}$ (as will follow below, it does not coincide with the real and imaginary parts of the

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function A_{p+1} in the expansion of the complex phase of the field in Refs. 1 and 4). Then f takes on the meaning of an aberrational function of zero order and the integrand in Eq. 1 can be represented in the form

$$d\xi/dz = \sum_{p=1}^{N} (dA_p/dz) \prod_{q=1}^{p} F_q (M_{q-1}(\xi;\ \alpha_{q-1})),$$

where $F_q(M_q-1)$ are positive definite monotonic functions ensuring the indicated property of A_p, and the arguments are defined recursively using integrals:

$$M_0(\xi;\ 0) = \xi, \quad M_1(\xi;\ \alpha_1) = \alpha_1 - \int_{\eta_0}^{\xi} d\gamma/F_1(\gamma),$$

$$M_n(\xi;\ \alpha_n) = \alpha_n + \int_{\eta_{n-1}}^{\nu_{n-1}} d\gamma/F_n(\gamma), \tag{2}$$

$$\nu_{n-1} = M_{n-1}(\xi;\ \alpha_{n-1}), \quad (n = 2,\ 3,\ \ldots,\ N-1),$$

$\alpha_n(z) = A_n(z) + C_n$, $C_n = \{C_q\}$ is a fixed vector in the space of the n aberrational functions; C_q ($1 \le q \le n$) are arbitrary constants; η_{n-1} is the greatest zero of the function $\gamma/F_n(\gamma)$ ($n = 1, 2, \ldots, N-1$).

Expression (1) allows one, using the method of characteristics and taking into account the representation of $d\xi/dz$, to find both integrals of the transfer equation

$$\partial I/\partial z + \nabla_\perp (I \nabla_\perp s) + \delta I = 0 \tag{3}$$

($\nabla_\perp = (\partial/\partial x, \partial/\partial y)$) for the intensity I of a bean field in a dissipative medium, characterized by an absorption coefficient δ, in the following form:

$$M_N(\xi;\ \alpha_N) = \text{const}_1, \tag{4}$$

$$I f^2 \xi \exp(\delta z) \cdot \prod_{q=1}^{N} F_q (M_{q-1}(\xi;\ \alpha_{q-1})) = \text{const}_2, \tag{4a}$$

where M_N is given by formula (2) with $n = N$. According to formula (2)

$$\partial M_n(\xi;\ \alpha_n)/\partial\xi = -\prod_{q=1}^{n} F_q^{-1} (M_{q-1}(\xi;\ \alpha_{q-1}))$$

$$(n = 1,\ 2,\ \ldots,\ N), \tag{5}$$

hence integral (4a) can be written as

$$I f^2 \xi \exp(\delta z)\ [\partial M_N(\xi;\ \alpha_N)/\partial\xi]^{-1} = -\text{const}_2. \tag{4b}$$

Determining the interrelationship between integrals (4) and (4b) at $z = 0$ for a beam with axial intensity I_0 and initial profile

$$I(\rho,\ 0) = I_0 \Phi(\xi_0^2)/P(0), \tag{6}$$

$\xi_0 = \xi(\rho,\ 0) = \rho/a_0$, $f(0) = 1$, we are led to the law

$$I(r) = I_0[\Phi(\Xi_N^2(\xi;\ \alpha_N;\ \alpha_{NO}))/\Phi(0)]\exp(-\delta z) \times$$

$$\times f^{-2}\left[\partial\Xi_N^2(\xi;\ \alpha_N;\ \alpha_{NO})/\partial(\xi^2)\right], \tag{7}$$

where the notation

$\Xi_N(\xi; \alpha_N; \alpha_{NO}) = M_N^{-1}(M_N(\xi; \alpha_N); \alpha_{NO})$ is used for the function $\xi = M_N^{-1}(\text{const}_1; \alpha_N)$, the inverse of the function on the left side of Eq. (4) taken at $\alpha_{NO} = \alpha_N(0)$, and the identically valid equality $M_N(\Xi_N(\xi; \alpha_N; \alpha_{NO}); \alpha_{NO}) = M_N(\xi; \alpha_N)$ is taken into account.

Let us turn now to the parabolic equation for the wave eikonal

$$2\partial s/\partial z + (\nabla_\perp s)^2 = \varepsilon'_{nl}[I]/\varepsilon_0 +$$

$$+ \left[\nabla_\perp^2 \ln I + \frac{1}{2}(\nabla_\perp \ln I)^2\right]/2k^2 \tag{8}$$

in which the real nonlinear additional term ε'_{n1} in the medium dielectric constant (ε_0 is its unperturbed value) depends on the intensity distribution of the beam, and $k = 2\pi/\lambda$. Substituting expression (1), which with the help of Eq. (5) takes the form

$$s(r) = s_0(z) + \rho^2 f'/2f + a_0^2 f^2 \sum_{p=1}^{N} W_p(\xi;\ \alpha_{p-1}) A'_p, \tag{1a}$$

in Eq. (8) and cancelling out like terms, we find

$$2\partial s/\partial z + (\nabla_\perp s)^2 = 2s'_0 + a_0^2 \xi^2 f f'' +$$

$$+ 2a_0^2 \sum_{p=1}^{N} W_p(\xi;\ \alpha_{p-1}) \cdot (f^2 A'_p)' +$$

$$+ 2a_0^2 f^2 \sum_{p=1}^{N} \sum_{q=1}^{p-1} \frac{\partial}{\partial\alpha_q} W_p(\xi;\ \alpha_{p-1}) A'_p A'_q +$$

$$+ a_0^2 f^2 \left[\sum_{p=1}^{N} \frac{\partial}{\partial\xi} W_p(\xi;\ \alpha_{p-1}) A'_p\right]^2, \tag{9}$$

where

$$W_p(\xi;\ \alpha_{p-1}) = -\int_0^{\xi} \left[\partial M_p(\zeta;\ \alpha_p)/\partial\zeta\right]^{-1} d\zeta. \tag{10}$$

In Eq. (9) and everywhere below the equality $\sum_{i=j}^{k} c_i \equiv 0$ is assumed to hold for $k < j$.

Since one can choose the new independent variable to be not more than the second power of the radial coordinate if it is not to introduce irrationality in the transformation of Eqs. (3) and (8), the expansion of the eikonal of the axially symmetric beam in series (1a) should contain only even powers of ρ. Hence, relating the order p of the function A_p with the lowest power $\rho^{2(p+1)}$ of the aberrational distortions of the wavefront corresponding to it, we conclude that the asymptotic form of the function $W_p(\xi;\ \alpha_{p-1})$ entering into Eq. (1a) as $\xi - 0$ is $\xi^{2(p+1)}$, and integrals (2) in accordance with Eq. (10) can be represented as

$$M_p(\xi;\ \alpha_p) = \alpha_p + \zeta^{-2p}\cdot R_{p-1}(\zeta^2;\ \alpha_{p-1})$$

$$(p = 1,\ 2,\ \ldots,\ N), \tag{2a}$$

where $R_p(x;\ \alpha_p)$ is some in general irrational function of its arguments, and $R_p(0;\ \alpha_p) = 1$. Since the aberrations $\rho^{2(p+1)}$ are characterized only by A'_p (see Eg. (1a)), asymptotic form (2a) remains the same as for $\alpha_{p-1} \equiv 0$, so that in addition $R_p(x;\ 0) = 1$.

Substitution of Eq. (2a) in Eq. (2) and comparison of both sides of the equalities as $\xi - 0$ give

$$\zeta^{-2} = \int_{\eta_0}^{\zeta} d\gamma/F_1(\gamma), \qquad \zeta^{-2n} = \int_{\eta_{n-1}}^{\zeta^{-2(n-1)}} d\gamma/F_n(\gamma)$$

$$(n = 2,\ 3,\ \ldots,\ N-1).$$

By differentiating the first equality, we find $F_1(\gamma) = (\gamma^3)/2$ and $\eta_0 = \infty$, and from the second it follows that $F_n(\gamma) = \left(1 - \dfrac{1}{n}\right)\cdot\gamma^{-1/(n-1)}$ and $\eta_{n-1} = 0$. This completes the determination of the integrals in Eq. (2):

$$M_1(\zeta;\ \alpha_1) = \alpha_1 + \zeta^{-2},$$

$$M_n(\zeta;\ \alpha_n) = \alpha_n + M_{n-1}^{\frac{n}{n-1}}(\zeta;\ \alpha_{n-1}), \tag{2b}$$

and the recursively defined function definition (11) (this fact will be used below). Note that inversion of formulas (16)—(17) leads to a series expansion of the exponential function of an infinite series and the combination of formulas (14)—(17) enables one to obtain the expansion of any other function of an infinite series.

The intensity (7), according to Eqs. (3) and (8), does not depend on the initial value $A_{NO} = A_N(0)$, and it is necessary to set $\alpha_{NO} = 0$ in Eq. (7) and choose the fixed vector in $\alpha_N(z)$ on the basis of the condition $C_N = -A_{NO}$. Upon inverting the integral (2a) at $p = N$ with respect to $\zeta = \xi$, taking relation (11) into account, and specifying the thusly formed function $\Xi_N(\xi;\alpha_N) \equiv \Xi_N(\xi;\alpha_N;0)$ in the profile (7) by setting $\alpha_N(z) = A_N - A_{NO}$, we find

$$\Xi_N(\xi;\ \alpha_N) = \xi^2 R_N^{-1/(N+1)}(\xi^2;\ \alpha_N). \tag{18}$$

Finally, combining formulas (2a), (2b), and (18), we establish the relationship

$$M_p(\xi;\ \alpha_p) = \Xi_p^{-2p}(\xi;\ \alpha_p)\ (p = 1,\ 2,\ \ldots,\ N), \tag{19}$$

and combining (11) and (18) we obtain

$$\Xi_N^{-2N}(\xi;\ \alpha_N) = \alpha_N + \Xi_{N-1}^{-2N}(\xi;\ \alpha_{N-1}) = \ldots\ . \tag{20}$$

Expression (18) allows us with the help of Eqs. (14)–(17) to expand the profile (7) in a series:

$$I(r) = I_0 e^{-\delta z} f^{-2}\left[1 + \sum_{m=1}^{\infty} D_N^{(m)}(d_m)\xi^{2m}\right]; \tag{7a}$$

$$D_N^{(m)}(\alpha_m) = Q_m + K_N^{(m-1)}(\alpha_{m-1}) + G_N^{(m)}(d_m) +$$

$$+ \sum_{j=1}^{m-1}\left[Q_{m-j} + K^{(m-j-1)}(\alpha_{m-j-1})\right]G_N^{(j)}(\alpha_j);$$

$$K_N^{(m-1)}(\alpha_{m-1}) = \sum_{j=1}^{m-1} Q_{m-j} R_N^{(j)}\left(-\frac{m-j}{N+1};\ \alpha_j\right);$$

$$G_N^{(m)}(\alpha_m) = (m+1)R_N^{(m)}\left(-\frac{1}{N+1};\ \alpha_m\right),$$

where the numbers Q_m determine the form of the initial profile of the beam (6), $\Phi(\xi_0^2) = \Phi(0)\left(1 + \sum_{m=1}^{\infty} Q_m \xi_0^{2m}\right)$. In particular, for the

$$R_p(x;\ \alpha_p) = \left[R_{p-1}(x;\ \alpha_{p-1}) + \alpha_p x\right]^{1+\frac{1}{p}},$$

$$R_0(x;\ 0) = 1, \tag{11}$$

whose Maclaurin series expansion

$$R_p(x;\ \alpha_p) = 1 + \sum_{m=1}^{\infty} R_p^{(m)}(\alpha_m)x^m \tag{12}$$

is given in terms of readily calculated coefficients

$$R_p^{(m)}(\alpha_m) = \begin{cases} 0, & p \le 0 \text{ or } m \le 0, \\ R_p^{(m)}(\alpha_m), & 1 \le m \le p, \\ R_p^{(m)}(\alpha_p), & 1 \le p \le m. \end{cases} \tag{13}$$

With the help of Eq. (13) the coefficients of the series expansion of the μth power of this function ($\mu \lessgtr 0$),

$$R_p^{\mu}(x;\ \alpha_p) = 1 + \sum_{m=1}^{\infty} R_p^{(m)}(\mu;\ \alpha_m)x^m, \tag{14}$$

are also expressed by recursion relations:

$$R_p^{(m)}(\mu;\ \alpha_m) = \mu R_p^{(m)}(\alpha_m) + \frac{1}{m}\sum_{j=1}^{m-1}\left[\mu(m-j)-j\right] \times$$
$$\times\ R_p^{(m-j)}(\alpha_{m-j}) R_p^{(j)}(\mu;\ \alpha_j). \tag{15}$$

Assuming in Eqs. (14)–(15) that $R_p^{(m)}(\mu; \alpha_m) = \mu L_p^{(m)}(\alpha_m)$ and taking the logarithm of Eq. (14) and then the limit $\mu - 0$, we obtain, taking Eq. (12) into account, the series expansion of the natural logarithm of the function $R_p(x;\ \alpha_p)$

$$\ln\left[1 + \sum_{m=1}^{\infty} R_p^{(m)}(\alpha_m)x^m\right] = \sum_{m=1}^{\infty} L_p^{(m)}(\alpha_m)x^m, \tag{16}$$

where

$$L_p^{(m)}(\alpha_m) = R_p^{(m)}(\alpha_m) - \frac{1}{m}\sum_{j=1}^{m-1} jR_p^{(m-j)}(\alpha_{m-j})L_p^{(j)}(\alpha_j) \tag{17}$$

The coefficients $R_p^{(m)}(\mu; \alpha_m)$ and $L_p^{(m)}(\alpha_m)$ possess property (13) and expansions (14) and (16) are valid (in the convergence regions) for arbitrary functions $R(x)$, irrespective of profile of the most general hyper-Gaussian form

$$\Phi(\xi_0^2) = \Phi(0)\ \exp\left[1 - \sum_{m=1}^{\infty} B_m\xi_0^{2m}\right], \tag{21}$$

Inverting Eq. (17), we obtain

$$Q_m = -B_m - \frac{1}{m}\sum_{j=1}^{m-1}(m-j)B_{m-j}Q_j \qquad (m \geq 1).$$

Expressions (19)–(20), from which it follows that

$$\partial\Xi_N/\partial\xi = \left(\partial\Xi_n/\partial M_p\right)\left(\partial M_p/\partial\xi\right)\ (p = 1,\ 2,\ \ldots\ N), \tag{22}$$

determine the nature of the interrelation between the eikonal and the beam intensity. According to Eq. (10) the field phase discontinuities caused by the local stationarity on the coordinate ξ of any $m(m \geq 1)$ of the N integrals (2a) occur in the z plane on circles of radius $\xi_p = \xi_p(z)$, given by the equations

$$\partial M_p(\xi;\ \boldsymbol{\alpha}_p)/\partial\xi = 0\ \ (p = p_j,\ j = 1,\ 2,\ \ldots,\ m),$$

and the field itself disappears on these circles according to Eq. (22) (see Eq. (7)). In other words, the beam disintegrates into m independent concentrically arranged parts which define its annual structure, each of which has in general its own aberrational distortions.

Bearing in mind that the parts of the split up beam can be described independently, we assume the following conditions to be fulfilled

$$\partial M_p(\xi;\ \alpha_p)/\partial\xi \neq 0\ \ (p = 1,2,\ldots N), \tag{23}$$

which ensure the continuous variation of the eikonal throughout the entire region occupied by the beam, except for its boundaries, where conditions (23) can be violated when $p = N$. Conditions (23) do not preclude a description of the annular structure of the beam if the field between the rings does not fall to zero. When conditions (23) are fulfilled, the function (10) can be represented by a convergent series

$$W_p(\xi;\ \alpha_{p-1}) = \sum_{m=1}^{\infty} W_p^{(p+m)}(\alpha_{p-1})\xi^{2(p+m)}, \tag{24}$$

in which

$$W_p^{(p+m)}(\alpha_{p-1}) = \frac{1}{4p(p+m)}\begin{cases}1, & m = 1,\\ S_{p-1}^{(m-1)}(-1;\ \alpha_{m-1}), & m > 1\end{cases} \tag{25}$$

are calculated taking into account formulas (2a) and (12), using the coefficients $S_p^{(m)}(-1; \alpha_m)$ of the expansion

$$S_p^{-1}(x;\ \alpha_p) = 1 + \sum_{m=1}^{\infty} S_p^{(m)}(-1;\ \alpha_m)x^m, \tag{14a}$$

analogous to expansion (14), namely the expansion of the reciprocal power of the function

$$S_p(x;\ \alpha_p) = 1 + \sum_{m=1}^{\infty}\left(1 - \frac{m}{p+1}\right)R_p^{(m)}(\alpha_m)x^m,$$

defined by the recursion relations (15) with $\mu = -1$ and R replaced by S.

The given aberration theory takes account of the components up to the $2(N+1)$th power inclusive of the radial coordinate in the expansion of the nonlinear additional term in the dielectric constant of the medium:

$$\varepsilon'_{nl}[I] = \varepsilon'_{nl,0}(z) + \sum_{s=1}^{N+1}\varepsilon'_{2s}(z)\xi^{2s} \tag{26}$$

Substituting Eqs. (9) and (26) in Eq. (8), carrying out the necessary transformations, taking formulas (18), (21),. and (24) into account, and collecting coefficients of the same power in the coordinate ρ (or ξ), we arrive at an equation for the unknown functions s_0, f, and A_N:

$$s'_0 = \varepsilon'_{nl,0}/2\varepsilon_0 - \alpha_0^2u_1/R_{diff}^2\ f^2,$$

$$f'' = \varepsilon'_2/\varepsilon_0\alpha_0^2 f + \left[u_1^2 - 4\left(u_2 + v_{1,1}\right)\right]/R_{diff}^2\ f^3,$$

$$\sum_{q=1}^{p}\left\{W_q^{(p+1)}(\alpha_{q-1})\left(f^2A'_q\right)' + f^2\sum_{m=1}^{q-1}A'_m\ \times\right.$$

$$\times\left[\frac{\partial}{\partial\alpha_m}W_q^{(p+1)}(\alpha_{q-1})A'_q + 2q\ (p-q+2)W_m^{(q)}(\alpha_{m-1})\ \times\right.$$

$$\left.\left.\times\sum_{n=1}^{p-q+1}W_n^{(p-q+2)}(\alpha_{n-1})A'_n\right]\right\} = \varepsilon'_{2p+2}/\varepsilon_0\alpha_0^2 - \tag{27}$$

$$- \left(2R^2_{diffr} f^2\right)^{-1} \Big\{ 2(p + 2)\Big[u_{p+2} + \sum_{q=1}^{p+1} v_{p-q+2,q}\Big] -$$

$$- \sum_{q=1}^{p+1}\Big[u_{p-q+2} u_q + \sum_{m=1}^{p-q+1}\Big[2u_{p-q-m+2} v_{m,q} +$$

$$+ \sum_{n=1}^{p-q-m+1} v_{p-q-m-n+2,q} v_{n,m}\Big]\Big]\Big\}$$

(p = 1, 2, ..., N), where we have made use of the following notation:

$$u_p = p\Big[B_p + \Gamma^{(p)}_N (\alpha_p)\Big],$$

$$v_{p,q} = (p + q)\, B_p R^{(q)}_N \Big(-\frac{p}{N+1};\ \alpha_q\Big), \tag{28}$$

$\Gamma^{(p)}_N(\alpha_p)$ are the expansion coefficients of the function

$$\ln\Big[1 + \sum_{m=1}^{\infty} G^{(m)}_N (\alpha_m) x^m\Big] = \sum_{m=1}^{\infty} \Gamma^{(m)}_N (\alpha_m) x^m, \tag{16a}$$

calculated accordingto the recursion formula (17) (after replacing L by Γ and R by G). $R_{diff} = ka_0^2$ is the diffraction length of the beam.

In order to calculate the coefficients $\varepsilon'_{2s}(z)$ in Eqs. (26)–(27), it is necessary to specify the nonlinearity mechanism of the medium. For example, in the case of a medium with cubic nonlinearity $\varepsilon'_{nl} = \varepsilon^{(2)} I$, using expansion (7a), we find

$$\varepsilon'_{2s}(z) = \frac{a_0^2 \varepsilon_0 \exp(-\delta z)}{R^2_{nl} f^2} \operatorname{sign} \varepsilon^{(2)} \times$$

$$\times \begin{cases} 1, & s = 0, \\ D^{(s)}_N (\alpha_s), & s = 1, 2, \ldots, N+1, \end{cases} \tag{29}$$

where the nonlinear length R_{nl} is equal to $R_{nl} = a_0(\varepsilon_0 / |\varepsilon^{(2)}| I_0)^{1/2}$. For a medium with a thermal nonlinearity $\varepsilon'_{nl} = (d\varepsilon / dT)\cdot[T(\rho, z) - T(0, z)]$, caused by a temperature field $T(r)$ which is nonuniform along a cross section of the beam, similarly, by solving the heat conduction equation we obtain

$$\varepsilon'_{2s}(z) = \frac{a_0^2 \delta \varepsilon_0 \exp(-\delta z)}{R_{nl}} \operatorname{sign}\Big[\frac{\partial \varepsilon}{\partial T}\Big] \times$$

$$\begin{cases} 0, & s = 0, \\ 1, & s = 1, \\ s^{-2} D^{(s-1)}_N (\alpha_{s-1}), & s = 2, 3, \ldots, N+1\,. \end{cases} \tag{30}$$

For thermal blooming $R_{nl,th} = 4\varepsilon_0 \kappa |d\varepsilon / dT| I_0$, where κ is the coefficient of thermal conductivity of the medium.

The system of equations (27), like Eqs. (3) and (8), has two integrals of motion, independent of the longitudinal coordinate z (Ref. 6), namely, the integral of the total[1] power of the beam in a nondissipative medium

$$P_1 = \int I(\rho, z)\, d^2\rho \tag{31}$$

and the second integral

$$P_2 = \int\Big[(\nabla_\perp s)^2 I + k^{-2}(\nabla_\perp I^{1/2})^2 - \varepsilon_0^{-1} F_{nl}(I)\Big] d^2\rho \tag{32}$$

where the function

$$F_{nl} = \int_0^I \varepsilon'_{nl}(\gamma) d\gamma.$$

Substituting expression (7) in Eq. (31) and taking into account that the beam power $P_1 = \pi a_0^2 I_0 \Phi^{-1}(0) \int_0^\mu \Phi(\mu) d\mu' \mu = \xi^2_{lim}(0)$, is determined by its limiting initial radius $a_{lim}(0) = a_0 \xi_{lim}(0)$, at which $\Phi(\xi^2_{lim}(0)) = 0$, the current limiting radius $a_{lim}(z) = a_0 f(z) \xi_{lim}(z)$ is found from the equation $\Xi_N(\xi_{lim}(z);\ \alpha_N) = \xi_{lim}(0)$, or in accordance with relations (11) and (18)

$$a_{lim}(z) = a_0 f(z) \Big[\Big[\ldots\Big[\Big[a_0^{2N}/a^{2N}_{lim}(0) - \alpha_N\Big]^{\frac{N-1}{N}} -$$

$$- \alpha_{N-1}\Big]^{\frac{N-2}{N-1}} \ldots \alpha_3\Big]^{\frac{2}{3}} - \alpha_2\Big]^{\frac{1}{2}} - \alpha_1\Big]^{-\frac{1}{2}}.$$

The second integral (32) determines the relation between the functions f and A_N and their first derivatives. For example, in the case of a medium with cubic nonlinearity, this relation is given by the equation

$$\Big[(\mathcal{E}_1^{1/2} f)'\Big]^2 - \Big[(\mathcal{E}_1^{1/2} f)'\Big]^2_{z=0} = \frac{\mathcal{E}_{20}}{\mathcal{E}_{10}} - \frac{\mathcal{E}_2 f^2}{\mathcal{E}_1} +$$

$$+ \frac{(\operatorname{sign} \varepsilon^{(2)})(\mathcal{E}_3 - \mathcal{E}_{30} f^2)}{4R^2_{nl} f^2} + \frac{\mathcal{E}_{40} f^2 - \mathcal{E}_4}{R^2_{diff} f^2},$$

in which

$$\mathcal{E}_i(\alpha_N;\ A'_N) = \int_0^{\xi^2_{lim}(0)} h_i(\zeta;\ \alpha_N;\ A'_N) \frac{\Phi(\zeta)}{\Phi(0)} d\zeta$$

$$(i = 1, 2, 3, 4),$$

$$h_1(\zeta;\ \alpha_N) = \xi^2(\zeta;\ \alpha_N),$$

$$h_2(\zeta;\ \alpha_N;\ A'_N) = \xi^2(\zeta;\ \alpha_N)\int_0^\zeta \xi^2(\zeta';\ \alpha_N) \times$$

$$\times \left[\frac{\partial}{\partial z}\ln\frac{\xi(\zeta';\alpha_N)}{\xi(\zeta;\ \alpha_N)}\right]^2 \frac{\Phi(\zeta')}{\Phi(0)}\,d\zeta',$$

$$h_3(\zeta;\ \alpha_N) = \left[\xi(\zeta;\ \alpha_N)\frac{\partial\xi}{\partial\zeta}\right]^{-1}\frac{\Phi(\zeta)}{\Phi(0)},$$

$$h_4(\zeta;\ \alpha_N) = \frac{1}{4}\left[\frac{\partial\xi(\zeta;\ \alpha_N)}{\partial\zeta}\right]^{-2}\left[\frac{\partial}{\partial\zeta}\ln\left[\frac{\xi}{\Phi(\zeta)}\frac{\partial\xi}{\partial\zeta}\right]\right]^2,$$

$g_{i0} = g_i(0; A'_N(0))$, and the function $\xi(\Xi_N^2; \alpha_N)$ is determined after inverting Eq. (18). Equation (33) generalizes the first integral of the equation for the dimensionless width of a Gaussian beam $f(z) = a(z)/a_0$ in nonaberrational theory[6] and transforms into the latter when nonlinear aberrations are negligible ($g_1 = g_3 = g_4 = 1$, $g_2 = 0$).

For an illustration. let us consider the description of wave aberrations up to the 6th order ($N = 2$), limiting ourselves to beams with Gaussian profile ($B_1 = 1$; $B_m = 0$; $m \geq 1$). Using formulas (14), (14a)–(17), (25), and (28)–(30) to calculate the coefficients $W_p^{(p+m)}(\alpha_{p-1})$, u_p, and $v_{p,q}$, and substituting them in Eq. (27), we arrive at the following system:

$$f'' = \frac{1}{f^3}\left(\mp\frac{\delta^\sigma U_0 e^{-\delta z}}{R_{nl}^{2-\sigma}} + \frac{V_0}{R_{diff}^2}\right),$$

$$(f^2A_1')' = \frac{1}{f^2}\left(\pm\frac{\delta^\sigma U_1 e^{-\delta z}}{R_{nl}^{2-\sigma}} - \frac{4V_1}{R_{diff}^2}\right), \qquad (27a)$$

$$(f^2A_2')' + 3(fA_1')^2 = \frac{1}{f^2}\left(\mp\frac{2\delta^\sigma U_2 e^{-\delta z}}{3^\sigma R_{nl}^{2-\sigma}} + \frac{12V_2}{R_{diff}^2}\right),$$

where

$$U_0 = f^{2\sigma}(1 + 2\alpha_1)^{1-\sigma},$$

$$U_1 = f^{2\sigma}(1 + 2\alpha_1)^\sigma\{2 + 6[2\alpha_1(1 + \alpha_1) - \alpha_2]\}^{1-\sigma};$$

$$U_2 = f^{2\sigma}\{1 + 3[2\alpha_1(1 + \alpha_1) - \alpha_2]\}^\sigma \times$$

$$\times \{1 + 12[\alpha_1(1 + \alpha_1)(1 + 2\alpha_1) - (1 + 3\alpha_1)\alpha_2]\}^{1-\sigma};$$

$$V_0 = 1 + 12[\alpha_1(1 + \alpha_1) - \alpha_2],$$

$$V_1 = 2\alpha_1[2 + 5\alpha_1(3 + 2\alpha_1)] - 3(5 + 22\alpha_1)\alpha_2;$$

$$V_2 = 2\alpha_1^2[5 + 14\alpha_1(2 + \alpha_1)] - 3[1 + 4\alpha_1(7 + 16\alpha_1) -$$

$$-11\alpha_2]\alpha_2,$$

and the value $\sigma = 0$ corresponds to the case of a medium with cubic nonlinearity, and $\sigma = 1$ corresponds to thermal blooming. The upper signs in Eq. (27a) are taken when $\varepsilon^{(2)} > 0$ $\left(\frac{d\varepsilon}{dT} > 0\right)$, and the lower ones — when $\varepsilon^{(2)} < 0$ $\left(\frac{d\varepsilon}{dT} < 0\right)$.

Neglecting aberrations ($A_1 = A_2 = 0$), the first equation of system (27a), for $\sigma = 0$, transforms into an equation for the dimensionless beam width in a cubic medium[6] and, for $\sigma = 1$, into an equation for the same function for the case of thermal blooming.[7] Taking only spherical aberrations ($A_1 \neq 0$, $A_2 = 0$) into account, the first two equations of system (27a) for $\sigma = 0$ coincide with the equations for the functions $f(z)$ and $\beta(z)$ of the aberration theory of the same kind.[5] As for the function $f(z)$ in the aberration theory, it cannot be given such an obvious interpretation as in the nonaberrational approximation, nor .can it be determined by a function that characterizes the variation of the intensity on the beam axis.

An analysis of equations (27a) has shown that for self-focusing, $\left(\varepsilon^{(2)}, \frac{d\varepsilon}{dT}\right) > 0$, aberrational beam distortions develop more rapidly in a medium with cubic nonlinearity than in that with a thermal one; while for defocusing, $\left(\varepsilon^{(2)}, \frac{d\varepsilon}{dT}\right) < 0$, on the contrary, the aberrations that arise for the greater extent remain in thermal blooming. Hence, the wave properties of a beam with regard to the influence of aberrations are manifested in nonlinear media of these two types in a different way, in spite of an identical mathematical description.

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