Differential and statistical invariants of a wavy surface. Part 1. General properties of invariants

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The transformation and symmetry properties of surface derivatives, correlation functions, and spectral moments at rotation of coordinate axes are examined. The spectral moments and derivatives of the same order are shown to be transformed in the same manner and can be represented in terms of their rotational invariants. Recurrence relations are found for invariants of different orders. The properties of isotropic, homogeneous, and inhomogeneous surfaces are discussed.

Introduction

When solving problems of oceanology and ocean physics associated with variability of sea roughness under the effect of various factors, adequate theoretical description of a wavy surface as a main object of the study becomes of primary importance. This description involves such basic issues as symmetry and invariance of differential and statistical characteristics of the sea roughness that call for consideration in more detail in connection with development of remote sensing methods.

As regards the differential characteristics of roughness (derivatives of surface disturbances), the differential geometry considers only the first and second quadratic forms of the surface,[1] that is, the properties of no higher than the second derivatives. As for the statistical characteristics, the overwhelming number of investigations made by now are limited to the study of wave height and slope statistics, that is, no higher than the second spectral moments. The well-known theoretical analysis of these characteristics is considered in the Longuet-Higgins paper.[2]

At the same time, in that paper and in some others (see, for example, Refs. 3 and 4) statistical distributions connected with radiation reflection from the surface are obtained, that may include derivatives up to the third order and form the basis for remote methods of sea roughness diagnostics. Using the remote optical methods, it is possible to study spectral moments up to the sixth order, and solve a wide range of scientific and applied problems. In this connection, analysis of invariant properties of higher (higher than the second) surface derivatives and spectral moments is rather urgent. Invariants of surface fields, that is, functions of these fields independent of the azimuth angle, are of particular interest in sea investigations. On the one hand, measurement of invariants is a convenient method for studying external factors and processes disturbing the sea surface. On the other hand, what is no less important, they allow the angular dependence (anisotropy) of roughness characteristics to be revealed in the explicit form. It should be noted that invariant properties of the surface are inseparably linked with its symmetry, and in this case we can speak about the properties of internal (point or local) and external (structural) symmetry, as it is conventional, for example, in crystal physics.[6]

In other words, if in the first case we consider the invariance with respect to point symmetry transformations (rotation, reflection, and inversion), then in the second case we take into account extra requirements connected with a choice of a particular physical model of the surface (isotropy, homogeneity, etc.). As a rule, the requirements of external symmetry lead to degeneration of invariants, at which some of them vanish. This simplifies the structure of fields (decreases the number of independent components) and simultaneously imposes certain restrictions on the shape of the spectrum for the chosen model of the surface.

This paper presents general analysis of invariant properties of differential and statistical characteristics of a wavy surface. It successively considers the transformation of surface characteristics (surface derivatives, spectral moments, and derivatives of the correlation function) at rotation of coordinate axes and develops a unified approach to the description of single-parameter invariants of any order based on representation of the transformation as a sum of irreducible terms of the rotation group. For further investigations by optical methods, the paper presents the equations for invariants up to the sixth order and gives a brief analysis of symmetry properties of statistical characteristics for homogeneous, inhomogeneous, and isotropic surfaces. Specific properties of the Gaussian surface being of the most practical interest will be considered in the second part of the paper.

1. Transformation and invariants of surface derivatives

As the coordinate axes are turned through the angle φ, the relation between the old (x, y) and new

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(x', y') coordinates of surface points is described by the following equations:

$$x' = x\cos\varphi + y\sin\varphi;\; y' = -x\sin\varphi + y\cos\varphi. \quad (1)$$

Similar relations at rotation of the coordinate axes take place for the differential operators as well:

$$\frac{\partial}{\partial x'} = \cos\varphi\frac{\partial}{\partial x} + \sin\varphi\frac{\partial}{\partial y},$$
$$\frac{\partial}{\partial y'} = -\sin\varphi\frac{\partial}{\partial x} + \cos\varphi\frac{\partial}{\partial y}. \quad (2)$$

Introducing designations for derivatives of surface elevation $\zeta(x, y)$ in the old and new coordinates

$$\partial^{p+q}\zeta/\partial x^p\partial y^q \equiv \zeta_{pq},\;\; \partial^{p+q}\zeta/\partial x'^p\partial y'^q \equiv \zeta'_{pq} \quad (3)$$

and using Eqs. (2), in the general case we can write

$$\zeta'_{pq} = \sum_{r=0}^{p}\sum_{s=0}^{q}(-1)^r C_p^r C_q^s (\cos\varphi)^{p-r+s} \times$$
$$\times(-\sin\varphi)^{q-s+r}\zeta_{p+q-r-s,r+s}, \quad (4)$$

where C_n^m are binomial coefficients. It should be noted that just the same equations can be written for derivatives F_{pq} of some function of surface elevations and their derivatives, if F is the rotation invariant. The last means that at rotation of the coordinate axes, F transforms into itself and remains unchanged. Equations (4) apparently correspond to the simplest case $F \equiv \zeta(x, y)$.

Define the differential operators as

$$D_{pq} = \partial^{p+q}/\partial x^p\partial y^q,\quad D'_{pq} = \partial^{p+q}/\partial x'^p\partial y'^q, \quad (5)$$

then $F_{pq} \equiv D_{pq}F$, $F'_{pq} \equiv D'_{pq}F$, and in place of Eqs. (4) for the derivatives F'_{pq} we can write the similar operator equations. It is important for us that the equation for the operator D'_{pq} can be represented in the form of a product of lower-order operator equations. It follows herefrom that the same final result can be obtained through the use of different intermediate equations:

$$D'_{p+r,q+s} = D'_{pq}D'_{rs} = D'_{ps}D'_{rq}. \quad (6)$$

This property of the equations will be used below for derivation of recurrence relations connecting the invariants of different orders.

Writing $(p + q + 1)$ of equations (4) for ζ'_{pq} or D'_{pq} of the same order $p + q$, we can find all rotation invariants for them. It follows from Eqs. (2) that the first-order derivatives have a single invariant

$$(\zeta'^2_{10} + \zeta'^2_{01})^{1/2} = (\zeta^2_{10} + \zeta^2_{01})^{1/2} = \tan\theta, \quad (7)$$

where θ is the surface slope angle. The second-order derivatives are transformed according to the equations

$$\zeta'_{20} = \zeta_{20}\cos^2\varphi + \zeta_{11}\sin2\varphi + \zeta_{02}\sin^2\varphi,$$
$$\zeta'_{11} = \zeta_{11}\cos2\varphi + (1/2)(\zeta_{02} - \zeta_{20})\sin2\varphi, \quad (8)$$
$$\zeta'_{02} = \zeta_{20}\sin^2\varphi - \zeta_{11}\sin2\varphi + \zeta_{02}\cos^2\varphi$$

and have two independent invariants

$$\zeta_{20} + \zeta_{02} = 2u_0,\;\; \zeta_{20}\zeta_{02} - \zeta^2_{11} = K_0, \quad (9)$$

where u_0 and K_0 are the mean and total (Gaussian) curvatures of u and K at the horizontal points ($\theta = 0$). For arbitrary surface points, they are, generally speaking, joint two-parameter invariants of the first- and second-order derivatives[1]:

$$u = [(1 + \zeta^2_{01})\zeta_{20} - 2\zeta_{10}\zeta_{01}\zeta_{11} + (1 + \zeta^2_{10})\zeta_{02}]/2d^{3/2};$$
$$K = K_0/d^2,\;\; d = 1 + \tan^2\theta \quad (10)$$

and usually written in the form

$$u = (k_1 + k_2)/2,\; K = k_1k_2,$$

where k_1 and k_2 are the principal curvature values, $k_{1,2} = u \pm (u^2 - K)^{1/2}$. We can also define the differential curvature $v = (k_1 - k_2)/2$ and the curvature anisotropy $s = k_1/k_2$. In this case, every pair of invariants (k_1, k_2), (u, v) or (K, s) can be taken as independent parameters in statistical description of the surface curvature (see, e.g., Ref. 5). With allowance for the invariants (9), Eqs. (8) can be written in the form

$$\zeta'_{20,02} = u_0 \pm v_0\cos(2\varphi - \varphi_2),$$
$$\zeta'_{11} = -v_0\sin(2\varphi - \varphi_2), \quad (11)$$

where $v_0 = (u_0^2 - K_0)^{1/2}$ is the differential curvature at horizontal surface points and $\varphi_2 = \arctan[2\zeta_{11}/(\zeta_{20} - \zeta_{02})]$.

Continuing this process, we can show that the arbitrary-order derivatives can be expressed through rotation invariants by means of ordinary trigonometric transformations. From the viewpoint of the group theory, this representation corresponds to resolution of transformation equations (4) into a sum of irreducible terms of the rotation group. To reveal invariants, we should represent the products $(\cos\varphi)^p(-\sin\varphi)^q$ as sums of functions of multiple arguments. This gives the equations

$$(\cos\varphi)^p(-\sin\varphi)^q = \sum_{r=1}^{n+1}\beta^{(r)}_{pq}\begin{cases}\sin(2r-1)\varphi, & \text{for even } p,\\ \cos(2r-1)\varphi, & \text{for odd } p\end{cases} \quad (12a)$$

for odd values of $p + q = 2n + 1$,

$$(\cos\varphi)^p(-\sin\varphi)^q = \begin{cases}\sum_{r=1}^{n}\beta^{(r)}_{pq}\sin2r\varphi, & \text{for odd } p,q,\\ \alpha_{pq} + \sum_{r=1}^{n}\beta^{(r)}_{pq}\cos2r\varphi, & \text{for even } p,q\end{cases} \quad (12b)$$

for even values of $p + q = 2n$. Here α_{pq} and $\beta_{pq}^{(r)}$ are the coefficients, and for α_{pq} we can directly write the general equation:

$$\alpha_{pq} = \alpha_{qp} =$$
$$= (p!q!)/[2^{2n}\, n!(p/2)!\,(q/2)!] \quad (\text{even } p,\, q). \quad (13)$$

After substitution of Eqs. (12) in Eqs. (4), the odd-order derivatives ($p + q = 2n + 1$) can be represented as

$$\zeta'_{pq} = \begin{cases} \sum\limits_{r=1}^{n+1} \beta_{pq}^{(r)} I_{2n+1}^{(r)} \sin[(2r-1)\varphi - \varphi_{2n+1}^{(r)}], & \text{for even } p, \\ \sum\limits_{r=1}^{n+1} \beta_{pq}^{(r)} I_{2n+1}^{(r)} \cos[(2r-1)\varphi - \varphi_{2n+1}^{(r)}], & \text{for odd } p, \end{cases}$$

$$[I_{2n+1}^{(r)}]^2 = [\Omega_{2n+1}^{(r)}]^2 + [\omega_{2n+1}^{(r)}]^2,$$
$$\varphi_{2n+1}^{(r)} = \arctan[\omega_{2n+1}^{(r)}/\Omega_{2n+1}^{(r)}], \quad (14)$$

where $I_{2n+1}^{(r)}$ are invariants, in which the components $\Omega_{2n+1}^{(r)}$ and $\omega_{2n+1}^{(r)}$ are noninvariant linear combinations of derivatives including derivatives with transposed indices with respect to each other

$$\Omega_{2n+1}^{(r)} = \sum_{s=0}^{n} G_{2n+1}^{(r,s)} \zeta_{2n+1-2s,2s},$$

$$\omega_{2n+1}^{(r)} = (-1)^{r-1} \sum_{s=0}^{n} G_{2n+1}^{(r,s)} \zeta_{2s,2n+1-2s}. \quad (15)$$

In these equations, the coefficients $G_{2n+1}^{(r,s)}$ are integer numbers, which can be found from recurrence relations (see below).

For even order derivatives ($p + q = 2n$) we obtain

$$\zeta'_{pq} = \begin{cases} \alpha_{pq} U_{2n} + \sum\limits_{r=1}^{n} \beta_{pq}^{(r)} V_{2n}^{(r)} \cos[2r\varphi - \varphi_{2n}^{(r)}], & \text{for even } p, \\ \sum\limits_{r=1}^{n} \beta_{pq}^{(r)} V_{2n}^{(r)} \sin[2r\varphi - \varphi_{2n}^{(r)}], & \text{for odd } p, \end{cases}$$

$$[V_{2n}^{(r)}]^2 = [H_{2n}^{(r)}]^2 + [h_{2n}^{(r)}]^2,$$
$$\varphi_{2n}^{(r)} = \arctan[h_{2n}^{(r)}/H_{2n}^{(r)}], \quad (16)$$

where U_{2n} are invariants represented by the sum of derivatives with even indices p and q, while the components $H_{2n}^{(r)}$ and $h_{2n}^{(r)}$ in invariants $V_{2n}^{(r)}$ are invariant linear combinations of derivatives with respectively only even or odd indices:

$$U_{2n} = \sum_{s=0}^{n} G_{2n}^{(s)} \zeta_{2n-2s,2s};$$

$$H_{2n}^{(r)} = \sum_{s=0}^{n} A_{2n}^{(r,s)} \zeta_{2n-2s,2s},$$

$$h_{2n}^{(r)} = \sum_{s=0}^{n-1} B_{2n}^{(r,s)} \zeta_{2n-2s-1,2s+1}. \quad (17)$$

Here $G_{2n}^{(s)}$, $A_{2n}^{(r,s)}$, and $B_{2n}^{(r,s)}$ are integer numbers, which, as $G_{2n+1}^{(r,s)}$, are obtained automatically from recurrence relations.

Using the multicative properties (6) for the transformation equations, we can show that $[V_{2n}^{(1)}]^2 = U_{2n}^2 - 4R_{2n}$, and the invariants U_{2n} and R_{2n} meet the recurrence relations

$$R_{2n+2} = (D_{20}\, U_{2n})\,(D_{02}\, U_{2n}) - (D_{11}\, U_{2n})^2;$$
$$U_{2n} = (D_{20} + D_{02}) U_{2n-2} = (D_{20} + D_{02})^n U_0 =$$
$$= \sum_{r=0}^{n} C_n^r \zeta_{2n-2r,2r}, \quad (18)$$

where $U_0 \equiv \zeta(x, y)$. At the same time, U_{2n} and the components of the invariants $I_{2n+1}^{(r)}$ and $V_{2n}^{(r)}$ can be found from the recurrence relations

$$\Omega_{2n+1}^{(1)} = D_{10}\, U_{2n}, \quad \omega_{2n+1}^{(1)} = D_{01} U_{2n},$$
$$\Omega_{2n+1}^{(r+1)} = D_{10}\, H_{2n}^{(r)} - D_{01}\, h_{2n}^{(r)},$$
$$\omega_{2n+1}^{(r+1)} = D_{01}\, H^{(r)} + D_{10}\, h_{2n}^{(r)};$$
$$U_{2n+2} = D_{10}\, \Omega_{2n+1}^{(1)} + D_{01}\, \omega_{2n+1}^{(1)};$$
$$H_{2n+2}^{(r)} = D_{10}\, \Omega_{2n+1}^{(r)} - D_{01}\, \omega_{2n+1}^{(r)};$$
$$h_{2n+2}^{(r)} = D_{01}\, \Omega_{2n+1}^{(r)} + D_{10}\, \omega_{2n+1}^{(r)}, \quad (19a)$$

from which some others can be derived. In particular, we have

$$\Omega_{k+2}^{(r+1)} = (D_{20} - D_{02})\, \Omega_k^{(r)} - 2D_{11}\, \omega_k^{(r)},$$
$$\omega_{k+2}^{(r+1)} = 2D_{11}\, \Omega_k^{(r)} + (D_{20} - D_{02})\, \omega_k^{(r)} \quad (19b)$$

and quite the same relations for $H_{k+2}^{(r+1)}$ and $h_{k+2}^{(r+1)}$. Besides, all these components ($f = H, h, \Omega, \omega$) meet the equations

$$f_{k+2}^{(r)} = (D_{20} + D_{02}) f_k^{(r)}; \quad 1 \le r \le E[(k+1)/2], \quad (20)$$

where $E(z)$ is the integer part of z. Note that the analogous recurrence relation for U_{k+2} is found from Eq. (18).

Thus, the derivatives ζ_{pq} of any order can be uniquely represented as a sum (14) and (16) including only rotation invariants. The recurrence relations (18)–(20) allow invariants of any order to be sequentially found. Obviously, at substitution of D_{pq} or F_{pq} for ζ_{pq}, all invariants or their components can be written in the operator form for an arbitrary invariant function of surface roughness and its derivatives. Particular equations for the invariants up to the six order are given in the next section.

2. Invariants and symmetry of statistical characteristics

Consider the angular transformation of spectral moments m_{pq}. They are determined by the equation[2]:

$$m'_{pq} = \iint\limits_{-\infty}^{\infty} S(k'_x, k'_y)\, k'^p_x\, k'^q_y \mathrm{d}k'_x\, \mathrm{d}k'_y, \quad (21)$$

where $\mathbf{k}' \equiv (k'_x, k'_y)$ is the wave vector in the turned coordinate system; $S(\mathbf{k}')$ is the spectrum of surface elevations. At rotation of the axes, the following equations are fulfilled:

$$k'_x = k_x \cos\varphi + k_y \sin\varphi, \quad k'_y = -k_x \sin\varphi + k_y \cos\varphi;$$

$$S(\mathbf{k}')\, \mathrm{d}\mathbf{k}' = S(\mathbf{k})\, \mathrm{d}\mathbf{k}. \quad (22)$$

Therefore, Eq. (21) can be represented in the form

$$m'_{pq} = \sum_{r=0}^{p} \sum_{s=0}^{q} (-1)^r C_p^r C_q^s (\cos\varphi)^{p-r+s} (-\sin\varphi)^{q-s+r} \times$$
$$\times\, m_{p+q-r-s,r+s}\, . \quad (23)$$

Note that Eq. (23) can be obtained using Eqs. (4) and (6) with statistical definition of moments in place of Eq. (21): $m'_{p+q,r+s} = \langle \zeta'_{pr}\, \zeta'_{qs} \rangle$ (hereinafter the angular brackets denote statistical averaging).

In a particular case at $q = 0$, from Eq. (23) we get the equation

$$m'_{p0} \equiv m_p(\varphi) = \sum_{r=0}^{p} C_p^r (\cos\varphi)^{p-r} (\sin\varphi)^r\, m_{p-r,r}\, , \quad (24)$$

which determines the moments $m_p(\varphi)$ of a single-dimensional spectrum of the surface in the direction φ (Ref. 2).

Comparison of Eqs. (4) and (23) shows that the spectral moments and surface derivatives are transformed in the same manner at rotation of the axes. It follows from here that the moments can be expressed through rotation invariants as

$$m_{pq} = \begin{cases} \sum\limits_{r=1}^{n+1} \beta_{pq}^{(r)} I_{2n+1}^{(r)} \sin\,[(2r-1)\varphi - \varphi_{2n+1}^{(r)}], & \text{for even } p, \\ \sum\limits_{r=1}^{n+1} \beta_{pq}^{(r)} I_{2n+1}^{(r)} \cos\,[(2r-1)\varphi - \varphi_{2n+1}^{(r)}], & \text{for odd } p \end{cases} \quad (25a)$$

(odd moments: $p + q = 2n + 1$),

$$m_{pq} = \begin{cases} \alpha_{pq} U_{2n} + \sum\limits_{r=1}^{n} \beta_{pq}^{(r)} V_{2n}^{(r)} \cos\,[2r\varphi - \varphi_{2n}^{(r)}], & \text{for even } p,q, \\ \sum\limits_{r=1}^{n} \beta_{pq}^{(r)} V_{2n}^{(r)} \sin\,[2r\varphi - \varphi_{2n}^{(r)}], & \text{for odd } p,q \end{cases} \quad (25б)$$

(even moments: $p + q = 2n$).

In this case, for direction spectral moments $m_{p+q}(\varphi)$ we have

$$\beta_{p+q,0}^{(r)} = 2^{1-p-q}\, C_{p+q}^{r+n} \quad (p + q = 2n,\ 2n + 1). \quad (26)$$

The moment invariants $I_{2n+1}^{(r)}$, U_{2n}, and $V_{2n}^{(r)}$ are apparently determined by Eqs. (15), (17), and (18), in which m_{pq} should be substituted for ζ_{pq}. Using the recurrence relations (18)–(20) and substituting the variables, for invariants and their components up to the sixth order, we obtain

$$U_0 = m_{00};\ \Omega_1^{(1)} = m_{10};\ \omega_1^{(1)} = m_{01},$$
$$U_2 = m_{20} + m_{02};\ H_2^{(1)} = m_{20} - m_{02};\ h_2^{(1)} = 2m_{11};$$
$$\Omega_3^{(1)} = m_{30} + m_{12},\ \ \Omega_3^{(2)} = m_{30} - 3m_{12};$$
$$U_4 = m_{40} + 2m_{22} + m_{04};\ \ H_4^{(1)} = m_{40} - m_{04};$$
$$h_4^{(1)} = 2(m_{31} + m_{13});$$
$$H_4^{(2)} = m_{40} + m_{04} - 6m_{22};\ h_4^{(2)} = 4(m_{31} - m_{13}); \quad (27)$$
$$\Omega_5^{(1)} = m_{50} + 2m_{32} + m_{14},\ \Omega_5^{(2)} = m_{50} - 2m_{32} - 3m_{14},$$
$$\Omega_5^{(3)} = m_{50} - 10m_{32} + 5m_{14};$$
$$U_6 = m_{60} + m_{06} + 3(m_{42} + m_{24});$$
$$H_6^{(1)} = m_{60} + m_{42} - m_{24} - m_{06};$$
$$h_6^{(1)} = 2(m_{51} + 2m_{33} + m_{15});$$
$$H_6^{(2)} = m_{60} + 5m_{42} - 5m_{24} + m_{06};$$
$$h_6^{(2)} = 4(m_{51} - m_{15});$$
$$H_6^{(3)} = m_{60} - 15m_{42} + 15m_{24} - m_{06};$$
$$h_6^{(3)} = 6m_{51} - 20m_{33} + 6m_{15},$$

where the components $\omega_{2n+1}^{(r)}$ are omitted for brevity. We can write them, substituting, according to Eq. (15), m_{qp} for m_{pq} in the components $\Omega_{2n+1}^{(r)}$ and multiplying these equations by $(-1)^{r-1}$; for example,

$$\omega_3^{(1)} = m_{03} + m_{21},\ \ \omega_3^{(2)} = 3m_{21} - m_{03}.$$

Higher-order invariants can be found directly using the recurrence relations, if D_{pq} in them is considered as a formal operator acting on moment indices in such a way that the index summation rule is fulfilled:

$D_{pq}\, m_{rs} \to m_{p+r,q+s}$. Thus, for example, using Eqs. (20), (27), and this rule we can easily find

$$U_8 = (D_{20} + D_{02})U_6 = m_{80} + 4m_{62} + 6m_{44} + 4m_{26} + m_{08};$$
$$H_8^{(1)} = (D_{20} + D_{02})H_6^{(1)} = m_{80} + 2m_{62} - 2m_{26} - m_{08} \quad (28)$$

and so on. Note that, rigorously, D_{pq} as a differential operator can be applied only to correlation functions (see below).

According to Eq. (26), the moments of odd (even) orders keep their values at rotation through the angle $\Delta\varphi$ equal to 2π and π, respectively. This means that their lowest internal symmetry is different and characterized, following the commonly accepted terminology,[6] by the first- (second-) order rotation axis, where the axis order equals $2\pi/\Delta\varphi$. As known, odd moments are nonzero only for an inhomogeneous surface, for example, under the conditions of developing sea roughness, when the wave profile asymmetry is observed and $S(-\mathbf{k}) \neq S(\mathbf{k})$. However, at established (developed) roughness, the sea surface becomes homogeneous (odd moments and all invariants $I_{2n+1}^{(r)}$ vanish), the symmetry of the spectrum increases: $S(-\mathbf{k}) = S(\mathbf{k})$, and all even moments can be represented as

$$m_{pq} = (-1)^n D_{pq}\psi(x, y)|_0 \equiv (-1)^n \psi_{pq}(0),\ p + q = 2n, \quad (29)$$

where $\psi(\mathbf{r})$ is the correlation function;

$$\psi(\mathbf{r}) = \int_{-\infty}^{\infty} S(\mathbf{k}) \cos(\mathbf{kr})\, d\mathbf{k}. \quad (30)$$

Here $\mathbf{r} \equiv (x, y) = \mathbf{R}' - \mathbf{R}$, and $\mathbf{R}'$ and $\mathbf{R}$ are the coordinates of two surface points. At rotation, the even derivatives are transformed as $(-1)^n m_{pq}$, therefore the functions $\eta_{pq} = (-1)^n \psi_{pq}$ have the same invariants $U_{2n}(\eta_{pq})$ and $V_{2n}^{(r)}(\eta_{pq})$, which meet the recurrence relations

$$U_{2n} = (D_{20} + D_{02})U_{2n-2} = \sum_{k=0}^{n} C_n^k\, \eta_{2n-2k,2k};$$
$$H_{2n+2}^{(r)} = (D_{20} + D_{02})\, H_{2n}^{(r)},$$
$$h_{2n+2}^{(r)} = (D_{20} + D_{02})\, h_{2n}^{(r)},$$
$$H_{2n+2}^{(r+1)} = (D_{20} - D_{02})\, H_{2n}^{(r)} - 2D_{11}\, h_{2n}^{(r)},$$
$$h_{2n+2}^{(r+1)} = 2D_{11}\, H_{2n}^{(r)} + (D_{20} - D_{02})\, h_{2n}^{(r)},\ 1 \leq r \leq n. \quad (31)$$

Under the conditions of developing roughness, when the surface, as was mentioned above, is inhomogeneous (more exactly, quasi-homogeneous[7]), its statistical description acquires a local character, that is,

$$\psi \equiv \psi(\mathbf{R}, \mathbf{r}),\ \ S \equiv S(\mathbf{R}, \mathbf{k}),\ \ m_{pq} \equiv m_{pq}(\mathbf{R}). \quad (32)$$

In this case, it makes sense to determine the second correlation function

$$\Phi(\mathbf{R}, \mathbf{r}) = \int_{-\infty}^{\infty} S(\mathbf{R}, \mathbf{k}) \sin(\mathbf{kr})\, d\mathbf{k}, \quad (33)$$

then for the odd spectral moments we have

$$m_{pq}(\mathbf{R}) = (-1)^n \Phi_{pq}(\mathbf{R}, 0),\ \ p + q = 2n + 1. \quad (34)$$

At odd $(p + q)$, the functions $\zeta_{pq} = (-1)^n \times \Phi_{pq}$ apparently have invariants $I_{2n+1}^{(r)}(\zeta_{pq})$ meeting the recurrence relations

$$\Omega_{2n+3}^{(r)} = (D_{20} + D_{02})\, \Omega_{2n+1}^{(r)};\ \omega_{2n+3}^{(r)} = (D_{20} + D_{02})\, \omega_{2n+1}^{(r)};$$
$$\Omega_{2n+3}^{(r+1)} = (D_{20} - D_{02})\, \Omega_{2n+1}^{(r)} - 2D_{11}\omega_{2n+1}^{(r)},$$
$$\omega_{2n+3}^{(r+1)} = 2D_{11}\, \Omega_{2n+1}^{(r)} + (D_{20} - D_{02})\, \omega_{2n+1}^{(r)},\ \ 1 \leq r \leq n + 1. \quad (35)$$

Consider the isotropic surface, whose all characteristics are independent of the angle φ. With allowance for this fact, it follows from Eq. (25) that

$$H_{2n}^{(r)} = h_{2n}^{(r)} = \Omega_{2n+1}^{(r)} = \omega_{2n+1}^{(r)} = 0;$$
$$m_{pq} = m_{qp} = \alpha_{pq} U_{2n}\ \ (\text{even } p, q;\ p + q = 2n), \quad (36)$$

that is, all invariants, except for U_{2n}, are degenerated, therefore all the odd moments are zero, and only one among all the nonzero even moments of any order is linearly independent. Note that the isotropic surface is a particular (extreme) case of a surface. In this case, the spectrum and its nonzero moments have a higher symmetry or the symmetry of the Curie group, when the order of the rotation axis is infinite.[6] In the similar manner, we can consider the properties of symmetry and degeneration of invariants for any other surface models.

Conclusion

In this paper, the invariants of differential and statistical characteristics of a wavy sea surface are successively considered and a general approach to their description is developed. It has been shown that the derivatives of surface elevations and spectral moments are transformed in the same manner at rotation of coordinate axes and they can be uniquely represented as sums of only rotation invariants. The derivatives of the correlation function can be represented in the same way.

The recurrence formulae have been obtained that allow single-parameter invariants of any order to be found. The approach developed here can be used to consider joint (multiparameter) invariants depending on several surface fields simultaneously.

The results of this work form the theoretical prerequisites for rigorous justification of optical methods for remote sensing of a wavy sea surface.

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