EFFECT OF THE OUTER SCALE OF TURBULENCE ON THE RESOLUTION OF AN IMAGING SYSTEM OPERATING IN THE ATMOSPHERE

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Received October 17, 1996

Equations for the optical transfer functions (OTFs) are derived for imaging systems operating in the atmosphere that consider the finite outer scale of turbulence for short and long times of exposure. The effect of the outer turbulence scale has been analyzed on the form of these OTFs. The integral resolution of the optical system has been calculated for different values of the outer scale of turbulence. It is shown that the finite scale of turbulence, even though it is larger by an order of magnitude than Fried's radius, should be considered to estimate the system resolution.

INTRODUCTION

In the theory of imaging of incoherent objects under conditions of the turbulent atmosphere the optical transfer functions[1] (OTFs) are widely used because these functions are well suited as means for the estimation of the optical system operation. Long- and short- exposure OTFs, calculated for Kolmogorov's model of the refractive index fluctuation spectrum,[2] give insight into limitations imposed by the turbulence onto the image quality.[1,3] It is known, however, that Kolmogorov's spectrum model is unrealistic for high and low spatial frequencies. Lukin,[4,5] for example, studied the effect of nonzero inner scale of turbulence on the OTF. Lukin et al.[6] showed that the outer scale of turbulence affects the calculated characteristics of the image. At the same time, we can find rather common in the literature the statement that the OTF has low sensitivity to the form of the refractive index fluctuation spectrum in the low-frequency range and hence the consideration of the finite outer scale of turbulence in calculation of the OTF is not obligatory (see, for example, Ref. 1). It should be noted that experimental data are contradictory. Some investigators (for example, Busher[7]) point out the good agreement of calculations on the basis of Kolmogorov's spectrum with their experimental results, whereas the others (for example, Rousset et al.[8] and Wiziniwich et al.[9]) indicate the necessity of considering the spectrum model with the finite outer scale of turbulence. In accordance, for example, with the results of Ref. 10, the outer scale averaged over the path appears to be comparable with characteristic Fried's radius. In Refs. 11 and 12, it was experimentally established that in the surface layer of the atmosphere the outer scale of turbulence is commensurate with the altitude above the underlying surface.

An increase in the efficiency of operation of adaptive optical systems by way of consideration of outer scale finiteness in their design was demonstrated in Ref. 3.

In connection with this, it is of interest to analyze the OTF with consideration of the finite outer scale of turbulence and his effect, in particular, on the resolution of the optical system operating in the turbulent atmosphere.

1. EXACT CALCULATION OF THE OTF

The most widespread model of the atmospheric turbulence, including in the explicit form the outer scale L_0, is the von Karman model with the spatial spectrum of the form

$$\Phi_n(\varkappa) = 0.033\, C_n^2(\varkappa^2 + \varkappa_0^2)^{-11/6}\,, \tag{1}$$

where C_n^2 is the structural constant of fluctuations of the refractive index n and $\varkappa_0 = 2\pi/L_0$.

The exponential model

$$\Phi_n(\varkappa) = 0.033\, C_n^2\, \varkappa^{-11/3}(1 - \exp(-\varkappa^2/\varkappa_0^2)) \tag{2}$$

is also widely used.

It was shown in Ref. 14 that calculations of the optical characteristics using the first spectral model can be reduced to the second model.

Let us consider a plane wave that after passage through a layer of the turbulent atmosphere of thickness z is incident on a thin lens with the aperture diameter D_0. In analogy with the examination of Kolmogorov's spectrum of turbulence,[2] Fried's radius

$$r_0 = 0.185\left(\frac{\lambda^2}{C_n^2\, z}\right)^{3/5}, \tag{3}$$

is convenient to use, where λ is the wavelength. Fried[2] showed that under certain assumptions the long-exposure OTF $\langle\tau(\Omega)\rangle_{LE}$ of the system turbulent

0235-6880/97/07 490-05 $02.00

atmosphere–lens, that is, the OTF averaged over an ensemble of realizations, is expressed in terms of the plane wave structure function $D(\rho)$

$$\langle\tau(\Omega)\rangle_{\rm LE} = \tau_0(\Omega)\,\exp\left(-\frac{1}{2}\,D(\lambda\Omega)\right), \qquad (4)$$

where Ω is the angular frequency[1]; $D(\rho) = D_l(\rho) + D_\phi(\rho)$, D_l and D_ϕ are the structural functions of amplitude and phase fluctuations, respectively; $\tau_0(\Omega)$ is the OTF of the optical system without the atmospheric turbulence.[1] We calculated the structural function $D(\lambda\Omega)$ from the formula[1]

$$D(\rho) = 8\pi k^2 z \int\limits_0^\infty [1 - J_0(\varkappa\rho)]\,\Phi_n(\varkappa)\varkappa\,{\rm d}\varkappa\,, \qquad (5)$$

where J_0 is the Bessel function. Formula (5) for spectra (1) and (2), respectively, yields (see Ref. 15, formulas (8.486.16) and (6.631.1))

$$D(\lambda\Omega) = C_1\left(\frac{\Omega_2}{\Omega_1}\right)^{5/3}\times$$
$$\times\left[1 - C_2\left(\frac{\Omega}{\Omega_2}\right)^{5/6} K_{5/6}\left(2\pi\,\frac{\Omega}{\Omega_2}\right)\right], \qquad (6)$$
$$D(\lambda\Omega) = C_3\left(\frac{\Omega_2}{\Omega_1}\right)^{5/3}\times$$
$$\times\left[1 - {}_1F_1\left(-\frac{5}{6}\,;\,1;\,-\frac{1}{4}\left(2\pi\,\frac{\Omega}{\Omega_2}\right)^2\right) + C_4\left(\frac{\Omega}{\Omega_2}\right)^{5/3}\right], \qquad (7)$$

where $K_{5/6}$ is the five-sixth order McDonald function; ${}_1F_1$ is the confluent hypergeometric function; $\Omega_0 = D_0/\lambda$, $\Omega_1 = r_0/\lambda$, and $\Omega_2 = L_0/\lambda$ are the characteristic angular frequencies determining the shape of the OTF $\langle\tau(\Omega)\rangle_{\rm LE}$, and numerical coefficients C_i are $C_1 = 0.1725$, $C_2 = 4.5998$, $C_3 = 0.9646$, and $C_4 = 7.1644$.

Further we take advantage of the formula

$$\langle\tau(\Omega)\rangle_{\rm SE} = \tau_0(\Omega)\,\exp\left\{-\frac{1}{2}\left(D(\lambda\Omega) - \frac{1}{2}\,(\lambda\Omega)^2\,\sigma_a^2\right)\right\} \quad (8)$$

derived in Ref. 2 for the short-exposure OTF averaged over an ensemble of realizations without consideration of stochastic displacement of each instantaneous image, where the variance of the wave-front tilts σ_a^2 is given by the formula[16]

$$\sigma_a^2 = \frac{64}{D_0^2}\times$$
$$\times\int\limits_0^1 \frac{4}{\pi}\left[(3u - 2u^3)\sqrt{1-u^2} - \arccos u\right] D_\phi(uD_0)\,u\,{\rm d}u\,. \quad (9)$$

Now we consider that the structural function of the phase fluctuations $D_\phi(\rho)$ is expressed through the wave structural function, namely, $D_\phi(\rho) = \alpha D(\rho)$, where $\alpha = 1$ in the near zone ($D_0 \gg \sqrt{\lambda z}$) and $\alpha = 1/2$ in the far zone[1] ($D_0 \ll \sqrt{\lambda z}$). This gives us the possibility for using Eqs. (6) and (7) to calculate the tilt variance by formula (9) and after that – short-exposure OTF (8). This calculation was made numerically and its results are considered below. Although we call as exact the formulas given above, this is true only within the limits of assumptions[2] made in the derivation of formulas (4) and (8).

2. APPROXIMATION FORMULAS FOR THE OTF

Formulas (4) and (6)–(9) reduce the problem to consideration of a linear system with the complicated but known transfer function whose form is unambiguously specified by the characteristic frequencies Ω, Ω_1, and Ω_2. Under conditions of the Earth' atmosphere, $\Omega_0 \ll \Omega_2$ ($D_0 \ll L_0$) is most often realized. Because we are interested in the OTFs only for angular frequencies $\Omega < \Omega_0$, the condition $\Omega_0 \ll \Omega_2$ allows us to expand the special functions, entering into Eqs. (6) and (7), in series and to consider only the terms of the order of Ω_1/Ω_2.

These approximation formulas can then be used to calculate the wave-front tilt variance σ_a^2 by formula (9). As a result, we derived the following formulas for $\langle\tau(\Omega)\rangle_{\rm LE}$ and $\langle\tau(\Omega)\rangle_{\rm SE}$:

$$\langle\tau(\Omega)\rangle_{\rm LE} =$$
$$= \tau_0(\Omega)\,\exp\left\{-\,3.44\left(\frac{\Omega}{\Omega_1}\right)^{5/3}\left[1 - \beta\left(\frac{\Omega}{\Omega_2}\right)^{1/3}\right]\right\}, \quad (10)$$

$$\langle\tau(\Omega)\rangle_{\rm SE} = \tau_0(\Omega)\times$$
$$\times\exp\left\{-3.44\left(\frac{\Omega}{\Omega_1}\right)^{5/3}\left[1-\alpha\left(\frac{\Omega}{\Omega_0}\right)^{1/3} - \beta(1-\alpha)\left(\frac{\Omega}{\Omega_2}\right)^{1/3}\right]\right\}. \quad (11)$$

Here, $\beta = 1.4848$ for von Karman's spectrum (1) and $\beta = 1.1480$ for exponential spectrum (2).

As an applicability criterion of the examined approximation, it is natural to use the requirement that the ratio ε of the largest neglected term of the corresponding series to the smallest considered term be small. Thus, the approximation is applicable when the inequalities

$$\varepsilon = 3.6\left(\frac{D_0}{L_0}^{5/3}\right) \ll 1 \quad \text{for spectrum (1)}\,, \qquad (12)$$

$$\varepsilon = 4{,}1\left(\frac{D_0}{L_0}^{2}\right) \ll 1 \quad \text{for spectrum (2)} \qquad (13)$$

hold true.

We now analyze Eq. (10) for the long-exposure OTF. Its form coincides with the formula for the short-exposure OTF of the system with Kolmogorov's spectrum of the refractive index of fluctuations,[1–2] but now the role of the cutoff frequency Ω_0 in the exponential multiplier plays a quantity proportional to the angular frequency Ω_2. This is a consequence of the assumption made by us which in fact means that we considered the dependence of the finite outer scale of turbulence only on the wave-front tilts. Recall that the wave-front tilts also determine the difference between the long- and short-exposure OTFs.

It is interesting to note that for $L_0 \approx 26D_0$ and von Karman's spectrum (1), Eq. (10) completely coincides with the formula for the short-exposure OTF of the system with Kolmogorov's spectrum of refractive index fluctuations in the far zone, and this approximation ensures good accuracy ($\varepsilon \sim 10^{-2}$). This, in particular, indicates that even for sufficiently large relative values of the outer scale, its finiteness should be considered in the calculation of the characteristics of optical systems as much as long- and short-exposure OTFs should be distinguished as $L_0 \to \infty$. The difference between long- and short-exposure OTFs corresponding to different L_0 is illustrated by Fig. 1 for the particular case $D_0 = r_0$.

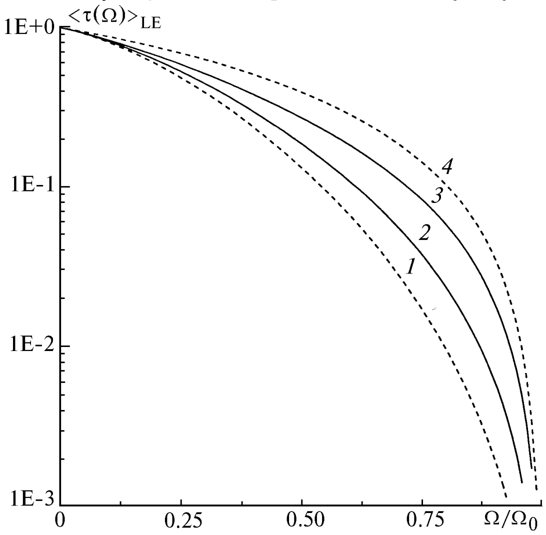

FIG. 1. Long-exposure OTF $\langle\tau(\Omega)\rangle_{LE}$ for $D_0/r_0 = 1$, $L_0 \to \infty$ (1), $L_0 = 50\ r_0$ (2), and $5r_0$ (3). The diffraction-limited OTF is shown by curve 4.

Analysis of Eq. (11) for the short-exposure OTF indicated that in the near zone ($\alpha = 1$) it coincides with the short-exposure OTF for systems with Kolmogorov's spectrum of the refractive index fluctuations.[1–2] Thus, the short-exposure OTF in the near zone, when conditions (12) and (13) hold true, in no way depends on the form of the fluctuation spectrum in the low-frequency range. The OTFs are identical for Kolmogorov's spectrum and for spectra (1) and (2). This is a consequence of the fact that when conditions (12) and (13) hold true, finiteness of L_0 affects only the wave-front tilts that do not affect the short-exposure OTF.

In the far zone ($\alpha = 1/2$), the dependence on the outer scale of turbulence is express on (11) holds true because now the finiteness of L_0 affects not only the large-scale phase fluctuations, but also the amplitude fluctuations resulting in additional (in comparison with the near zone) degradation of the image. In Fig. 2, the short-exposure OTF in the far zone is shown corresponding to different values of the outer scale of turbulence.

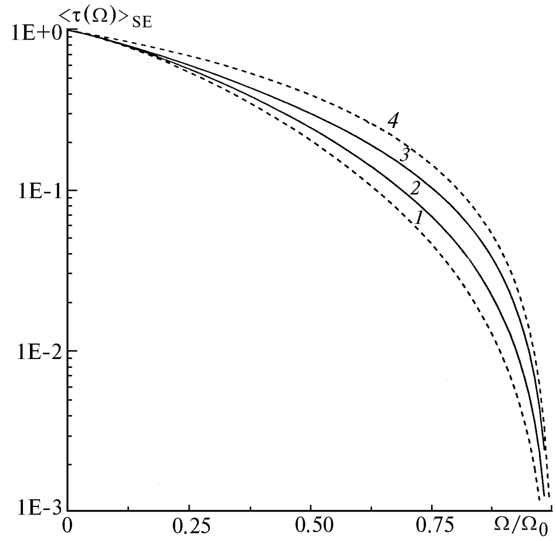

FIG. 2. Short-exposure OTF $\langle\tau(\Omega)\rangle_{SF}$ in the far zone for $D_0/r_0 = 1$, $L_0 \to \infty$ (1), $L_0 = 50\ r_0$ (2) and $5r_0$ (3). The diffraction-limited OTF is shown by curve 4.

Because the characteristic value of the outer scale is minimum on near-ground paths, just in this case the consideration of its finiteness is most important. Therefore, the calculation of the corresponding OTF for the case of a point light source located within the atmosphere is of interest (approximation of a spherical wave). In this case, the calculation analogous to that done in Ref. 17 for Kolmogorov's spectrum of the refractive index may be done only approximately for the case $P_0 \ll L_0$ and yields the formulas analogous to Eqs. (10) and (11). Now, however, the factor 3/8 appears in front of the exponential term in these formulas (as in case of Kolmogorov's spectrum) and the coefficient β is substituted by $8/9\beta$. Therefore, the OTF of the point source located within the atmosphere appears to be slightly less sensitive to the value of the outer scale of turbulence than the OTF of infinitely distant source (approximation of a plane wave).

3. EFFECT OF THE OUTER SCALE OF TURBULENCE ON THE OPTICAL SYSTEM RESOLUTION

One of the parameters most commonly used for the estimation of resolution of the optical system with the average OTF $\langle\tau(\Omega)\rangle$ is so-called integral resolution

$$R = 2\pi \int_0^\infty \langle \tau(\Omega) \rangle \, \Omega \, d\Omega \quad . \tag{14}$$

As known, the minimum resolved distance can be estimated as $\delta l = 1/2R^{1/2}$ (see Ref. 17). Without atmospheric turbulence, the resolution R depends only on the angular cutoff frequency of a lens, Ω_0. It can be easily shown that $R(\Omega_0) = \pi\Omega_0^2/4$. For Kolmogorov's spectrum of the refractive index fluctuations, R was calculated in Ref. 2. In Ref. 5, the corresponding calculation was done, considering the nonzero inner scale of the turbulence.

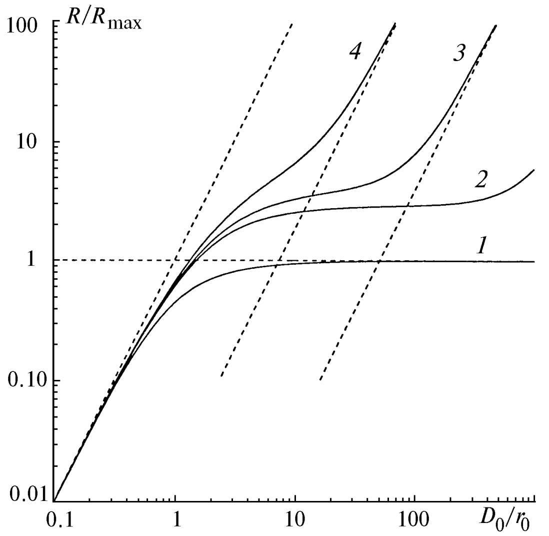

FIG. 3. Normalized integral resolution R/R_{max} for the long time of exposure as a function of D_0/r_0: $L_0 \to \infty$ (1), $L_0 = 20r_0$ (2), $15r_0$ (3), and $10r_0$ (4). The dashed curves show asymptotes (5).

To estimate the effect of the finite outer scale of turbulence on the integral resolution of the optical system, von Karman's specrtrum[1] is considered in the present paper. The calculation of integral (14) was done numerically using exact formulas (4), (6), (8), and (9). In this case, the variance of the wave front tilts σ_a^2 was also determined by numerical integration in accordance with Eq. (9). For definiteness, we further assume that $L_0 > r_0$.

First, we consider the long-exposure OTF. The results of calculation of R as a function of the ratio D_0/r_0 for different L_0 are shown in Fig. 3. The values of R are normalized by the parameter $R_{max} = \pi/(r_0/\lambda)^2$ being the maximum resolution in case of Kolmogorov's spectrum.[2] In this case, as seen from Fig. 3, the resolution saturates with the increase of D_0. The increase of the aperture diameter to the values larger than several r_0 no longer increases the image quality.

In the case of the finite outer scale of turbulence L_0 and small aperture diameters ($D_0 \ll r_0$), the turbulence insignificantly affects the resolution, which in this range of D_0 variation is determined mainly by the aperture diameter. However now, as seen from formulas (4) and (6), even for the infinitely high frequencies Ω, the OTF reduces to the finite nonzero quantity $\exp(-1/2C_1(\Omega_2/\Omega_1)^{5/3})$. For $D_0 >> r_0$, this leads to the increase of R as D_0 increases. All said above allows us to write down the asymptotic formula for R

$$R = \begin{cases} R_0(\Omega_0) \, , & D_0 << r_0 \, , \\ R_0(\Omega_0) \exp\{ -1/2 \, C_1 \, (\Omega_2/\Omega_1)^{5/3}\} \, , & D_0 >> L_0 \, . \end{cases} \tag{15}$$

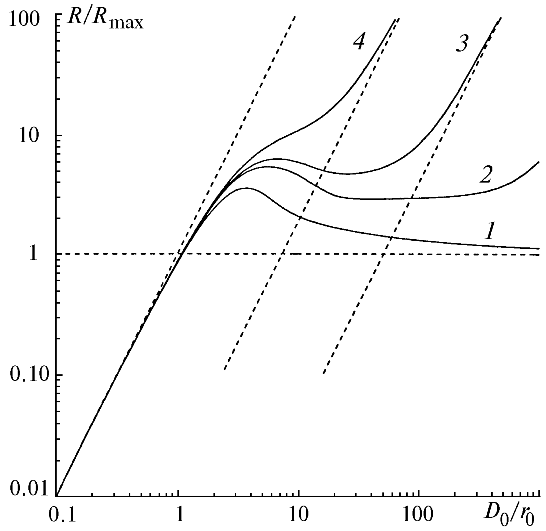

FIG. 4. Normalized integral resolution R/R_{max} for short time of-exposure in the near zone as a function of D_0/r_0: $L_0 \to \infty$ (1), $L_0 = 20r_0$ (2), $15r_0$ (3), and $10r_0$ (4). The dashed curves show asymptotes (15).

These asymptotes are shown in Fig. 3 by the dashed curves. Analogous calculation also was made for short-exposure OTF. In Fig. 4, the results for the near zone ($\alpha = 1$) are shown. For $D_0 \ll r_0$ and $D_0 \gg L_0$, the resolution calculated for the short-exposure OTF approaches the same asymptotes (15). For sufficiently large L_0, the integral resolution R has the characteristic local maximum at D_0 of the order of several r_0.

The absolute value of this maximum increases with the decrease of L_0. The value of D_0 at which it is attained also increases. For L_0 smaller than ~$12r_0$, this maximum disappears, and R monotonically increases with the increase of D_0. Thus, in this case the behavior of the resolution differs from predicted one for Kolmogorov's spectrum not only quantitatively but also qualitatively. We note that the values of L_0 of the order of $10r_0$ are real. For example, for the optical system operating along the Earth's surface at an altitude of man's height ($h \approx 2$ m), the outer scale can be estimated as $L_0 \approx h/2 = 1$ m. Then for $r_0 = 5$-- 10 cm, the ratio L_0/r_0 can take values from 20 to 10.

4. CONCLUSIONS

On the basis of analysis of the formulas derived here for the OTF of the optical system operating in the atmosphere with the finite outer scale of turbulence, the following conclusions can be drawn. First, the real consideration of finiteness of the outer scale is necessary for the explicit calculation of characteristics of the optical system in many practically interesting situations, as far as it results in different conclusions about the behavior of this optical system. Approximation formulas derived in the present paper can be used to estimate easily the effect of the finite outer scale of turbulence on the form of the OTF for $D_0 << L_0$ and to compare the long- and short-exposure OTFs. The comparison in combination with the calculation of the resolution by the exact formulas leads to the conclusion about much stronger effect of the finite outer scale on the long-exposure OTFs than on the sport-exposure OTFs, especially in the near zone, where the OTFs can be calculated by the formula for Kolmogorov's spectrum, given that conditions (12) and (13) are satisfied.

Our calculations indicate that consideration of the finite outer scale in imaging can be very complicated, because the characteristics of the system turbulent atmosphere–lens may differ substantially for different relationships of characteristic scales D_0, r_0, and L_0.

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