

ON THE THEORY OF OPTICAL TRANSFER OPERATORS OF THE ATMOSPHERE

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Received January 21, 1991*

The formula is analyzed for the horizontally inhomogeneous component of the brightness field of outgoing radiation above an isotropically reflecting surface. The dependence of this component of the radiation brightness upon the spatial variations of the surface albedo is shown to be nonlinear.

Both the direct and inverse optical transfer operators of the atmosphere were constructed in Refs. 1–3 which related the two-dimensional albedo of a Lambertian underlying surface illuminated by incident solar radiation to the brightness of radiation reflected by the surface-atmosphere system. The theory of the optical transfer operators of the atmosphere continued to develop in rather complex models which accounted for the horizontal inhomogeneity of the atmosphere, for the radiation polarization, and for the anisotropy of reflection from the Earth's surface.^{4–6} Such a theory has been already completed for the case of isotropic nonuniform reflection from the underlying surface, and it can be applied to the systems of numerical processing of satellite images. Therefore, new results (see Refs. 7–14) obtained in this field are of particular interest.

Let z be the vertical coordinate $\mathbf{r} = \{x, y\}$ be the vector of horizontal coordinates; $\mathbf{s} = \{\mu, \mathbf{s}_\perp\}$ be the unit vector of the direction $\mu = \cos\theta$; $\mathbf{s}_\perp = \sqrt{1 - \mu^2} \{\cos\varphi, \sin\varphi\}$; θ and φ be the zenith and the azimuth observation angles; $z = 0$ and $z = h$ be the lower and upper boundaries of the atmosphere; Ω be the unit sphere, Ω_+ and Ω_- be the upper and lower hemispheres; $q(\mathbf{r})$ be the inhomogeneous surface albedo; \bar{q} and $\tilde{q}(\mathbf{r}) = q(\mathbf{r}) - \bar{q}$ be the average albedo and its variation and, $f \equiv f(\mathbf{s}, \mathbf{s}')$, $\alpha(z)$, and $\sigma(z)$ be the scattering phase function and the coefficients of attenuation and scattering of light in the atmosphere.

On the basis of the model of a horizontally homogeneous atmosphere illuminated by the Sun and founded by the isotropically reflecting Lambertian surface, the brightness of the outgoing radiation has the form^{2,3}:

$$I = \bar{I} + \tilde{I}, \quad (1)$$

Here

$$\bar{I} = D + \bar{q}\bar{E}\Psi_0 \quad (2)$$

is the average component of brightness, D is the brightness of the haze; $\bar{E} = (E_0(1 - \bar{q}c_0))^{-1}$; πE_0 and Ψ_0 are the illuminance of the Earth's surface and the norm of the optical spatial-frequency characteristic of the atmosphere at $\bar{q} = 0$; c_0 is the spherical albedo of the atmospheric layer. The variation \tilde{I} can be represented in the form:

$$\tilde{I} = \sum_{k=1}^{\infty} \Phi_k, \quad (3)$$

where the "orders of multiplicity of reflections" Φ_k are evaluated by recursion from the system of boundary-value problems

$$\begin{cases} L\Phi_k = S\Phi_k; \Phi_k|_{z=0} = 0; \\ \Phi_k|_{z=h} = \bar{q}R\Phi_k + q(\mathbf{r})R\Phi_{k-1}, k \geq 1. \end{cases} \quad (4)$$

Here $L = (\mathbf{s}, \nabla) + \alpha(z)$ is the differential operator of radiative transfer; $S: S\Phi_k = \sigma(z) \int_{\Omega} \Phi_k f \, ds$ is the integral

operator of multiple scattering; $R: R\Phi_k = \frac{1}{\pi} \int_{\Omega_+} \hat{\Phi}_k|_{z=h} \mu \, ds$ is

the reflection operator.

The Fourier transform of the Eq. (3) may be represented in the form^{5,7}

$$\begin{aligned} \hat{I} &= \sum_{k=1}^{\infty} \Phi_k = \bar{E}W(\mathbf{p}) \sum_{k=1}^{\infty} Q^{k-1} \hat{q}(\mathbf{p}) = \bar{E}W(\mathbf{p}) \sum_{k=0}^{\infty} Q^k \hat{q}(\mathbf{p}) = \\ &= \bar{E}W(\mathbf{p}) [E - Q]^{-1} \hat{q}(\mathbf{p}), \end{aligned} \quad (5)$$

where $W(\mathbf{p}) = \Psi(\mathbf{p}) [1 - \bar{q}C(\mathbf{p})]^{-1}$; $\Psi(\mathbf{p})$ is the optical spatial-frequency characteristic of the atmosphere; $\mathbf{p} = \{p_x, p_y\}$ is the vector of spatial frequencies;

$\hat{q}(\mathbf{p}) = \int_{-\infty}^{\infty} q(\mathbf{r}) e^{i(\mathbf{p}, \mathbf{r})} d\mathbf{r}$ is the Fourier spectrum of the albedo;

E is the unit operator; Q is an operator acting upon a set of functions $\{g(\mathbf{p})\}$ following the rule:

$$Q g(\mathbf{p}) = \frac{1}{(2\pi)^2} \int_{-\infty}^{\infty} \hat{q}(\mathbf{p} - \mathbf{p}') H(\mathbf{p}') g(\mathbf{p}') d\mathbf{p}';$$

$$H(\mathbf{p}) = RW(\mathbf{p}) = C(\mathbf{p}) [1 - \bar{q}C(\mathbf{p})]^{-1}; C(\mathbf{p}) = R\Psi(\mathbf{p}).$$

Analytically, the sum of series (3) can be written in the following way:

$$\tilde{I} = \tilde{I}_1 + \tilde{I}_n = \frac{\bar{E}}{(2\pi)^2} \int_{-\infty}^{\infty} W(\mathbf{p}) \hat{q}(\mathbf{p}) e^{-i(\mathbf{p}, \mathbf{r})} d\mathbf{p} +$$

$$\begin{aligned}
 & + \bar{E} \sum_{k=2}^{\infty} \frac{1}{(2\pi)^{2k}} \int_{-\infty}^{\infty} \dots \int_{-\infty}^{\infty} W(\mathbf{p}) \bar{H}(\mathbf{p}_1) \dots \bar{H}(\mathbf{p}_{k-1}) \times \\
 & \times \hat{q}(\mathbf{p} - \mathbf{p}_1) \hat{q}(\mathbf{p}_1 - \mathbf{p}_2) \dots \hat{q}(\mathbf{p}_{k-2} - \mathbf{p}_{k-1}) \times \\
 & \times \hat{q}(\mathbf{p}_{k-1}) e^{-i(\mathbf{p}, \mathbf{r})} d\mathbf{p}_1 \dots d\mathbf{p}_{k-1} d\mathbf{p}, \tag{6}
 \end{aligned}$$

where \bar{I}_l, \bar{I}_n are the linear and nonlinear components of variations of the albedo.

Similar relations hold for the problem that takes into account the state of polarization.⁵ We have for the Stokes vector of the outgoing radiation

$$\mathbf{J} = \bar{\mathbf{J}} + \tilde{\mathbf{J}}. \tag{7}$$

Here

$$\bar{\mathbf{J}} = \mathbf{D} + \bar{q} \bar{E} \Psi_0 \tag{8}$$

is the "average" component of the Stokes vector; \mathbf{D} is the vector "haze"; Ψ_0 is the norm of the optical spatial-frequency vector characteristic of the atmosphere at $\bar{q} = 0$.

The variation $\tilde{\mathbf{J}}$ is given by the series

$$\tilde{\mathbf{J}} = \sum_{k=1}^{\infty} \Phi_k, \tag{9}$$

where Φ_k satisfies the system of recursive boundary-value problems

$$\begin{cases} L\Phi_k = S\Phi_k; \Phi_k \Big|_{s \in \Omega_+}^{z=0} = 0; \\ \Phi_k \Big|_{s \in \Omega_-}^{z=h} = \left[\bar{q} R \Phi_k + \tilde{q}(\mathbf{r}) R \Phi_{k-1} \right] \mathbf{1} \end{cases} \tag{10}$$

$S: S\Phi_k = \sigma(z) \int_{\Omega} P \Phi_k ds$ is the matrix operator of multiple scattering; P is the angular matrix; $\mathbf{l} = \{1, 0, 0, 0\}$.

Following the above-indicated scheme, we obtain^{5,8}:

$$\hat{\mathbf{J}} = \bar{E} W(\mathbf{p}) [E - Q]^{-1} \hat{q}(\mathbf{p}), \tag{11}$$

where $W(\mathbf{p}) = \Psi(\mathbf{p}) [1 - \bar{q} \bar{C}(\mathbf{p})]^{-1}$; $\Psi(\mathbf{p})$ is the optical spatial-frequency vector characteristic of the atmosphere; E is the unit matrix operator; Q is an operator which acts upon a set of vector functions $\{g(\mathbf{p})\} = \{g(\mathbf{p})\mathbf{l}\}$ following the rule

$$Q g(\mathbf{p}) = \frac{1}{(2\pi)^2} \int_{-\infty}^{\infty} \hat{q}(\mathbf{p} - \mathbf{p}') \bar{H}(\mathbf{p}') g(\mathbf{p}') d\mathbf{p}';$$

$$\bar{H}(\mathbf{p}) = \bar{C}(\mathbf{p}) [1 - \bar{q} \bar{C}(\mathbf{p})]^{-1}; \bar{C}(\mathbf{p}) = R \Psi_1(\mathbf{p});$$

$\Psi_1(\mathbf{p})$ is the first component of the vector function $\Psi(\mathbf{p})$. Analytically, the sum of the series (9) can be written as

$$\tilde{\mathbf{J}} = \tilde{\mathbf{J}}_1 + \tilde{\mathbf{J}}_n = \frac{\bar{E}}{(2\pi)^2} \int_{-\infty}^{\infty} \mathbf{W}(\mathbf{p}) \hat{q}(\mathbf{p}) e^{-i(\mathbf{p}, \mathbf{r})} d\mathbf{p} +$$

$$\begin{aligned}
 & + \bar{E} \sum_{k=2}^{\infty} \frac{1}{(2\pi)^{2k}} \int_{-\infty}^{\infty} \dots \int_{-\infty}^{\infty} \mathbf{W}(\mathbf{p}) \bar{H}(\mathbf{p}_1) \dots \bar{H}(\mathbf{p}_{k-1}) \times \\
 & \times \hat{q}(\mathbf{p} - \mathbf{p}_1) \hat{q}(\mathbf{p}_1 - \mathbf{p}_2) \dots \hat{q}(\mathbf{p}_{k-2} - \mathbf{p}_{k-1}) \times \\
 & \times \hat{q}(\mathbf{p}_{k-1}) e^{-i(\mathbf{p}, \mathbf{r})} d\mathbf{p}_1 \dots d\mathbf{p}_{k-1} d\mathbf{p}, \tag{12}
 \end{aligned}$$

where $\bar{\mathbf{J}}_l, \bar{\mathbf{J}}_n$ are the components of the vector $\bar{\mathbf{J}}$, linear and nonlinear for variations of the albedo.

On the basis of Eq. (5), the authors of Ref. 7 concluded that

$$\tilde{I} = \frac{\bar{E}}{(2\pi)^2} \int_{-\infty}^{\infty} \frac{W(\mathbf{p}) \hat{q}(\mathbf{p}) e^{-i(\mathbf{p}, \mathbf{r})} d\mathbf{p}}{1 - \frac{1}{(2\pi)^2} \int_{-\infty}^{\infty} \hat{q}(\mathbf{p} - \mathbf{p}') H(\mathbf{p}') d\mathbf{p}'}. \tag{13}$$

According to Ref. 8, Eq. (11) yields

$$\tilde{\mathbf{J}} = \frac{\bar{E}}{(2\pi)^2} \int_{-\infty}^{\infty} \frac{W(\mathbf{p}) \hat{q}(\mathbf{p}) e^{-i(\mathbf{p}, \mathbf{r})} d\mathbf{p}}{1 - \frac{1}{(2\pi)^2} \int_{-\infty}^{\infty} \hat{q}(\mathbf{p} - \mathbf{p}') H(\mathbf{p}') d\mathbf{p}'}. \tag{14}$$

Equations (13) and (14) represent, in closed form, the sums of the corresponding series (5) and (11). Taking into account the identity of formulas (13) and (14), a scalar problem may be considered, by way of example, in our further study.

We shall now expand the integral in the right side of Eq. (13) into a series

$$\begin{aligned}
 \tilde{I} = \tilde{I}_1 + \tilde{I}_n = & \frac{\bar{E}}{(2\pi)^2} \int_{-\infty}^{\infty} W(\mathbf{p}) \hat{q}(\mathbf{p}) e^{-i(\mathbf{p}, \mathbf{r})} \times \\
 & \times \left[1 + \sum_{k=1}^{\infty} \left(\frac{1}{(2\pi)^2} \int_{-\infty}^{\infty} \hat{q}(\mathbf{p} - \mathbf{p}') H(\mathbf{p}') d\mathbf{p}' \right)^k \right] d\mathbf{p}. \tag{15}
 \end{aligned}$$

It was demonstrated in Ref. 15 for the actual values of the atmospheric optical parameters and surface albedo that the contribution of the component \tilde{I}_n to the radiation brightness I is about 1%. The contribution may increase to a few per cent for large values of \bar{q} and $\max |\tilde{q}(\mathbf{r})|$ and for the slightly elongated scattering phase functions. The difficulty of calculating the component \tilde{I}_n from Eq. (6), on the one hand, and its small contribution to I , on the other, makes one to neglect the value \tilde{I}_n in practical applications. Meanwhile, formula (13) makes it possible to take into account the nonlinear component in a simple way and to eliminate the problem of its preliminary

estimation. However, attentive examination of this formula shows that it was ill-found. When going over from Eq. (5) to Eq. (13), an error of identifying the sum of the operator series with the sum of a geometric progression was made. Let us compare Eqs. (6) and (15). The linear terms in both

of them coincide $\tilde{I}_1 = \frac{\bar{E}}{(2\pi)^2} \int_{-\infty}^{\infty} W(\mathbf{p}) \hat{q}(\mathbf{p}) e^{-i(\mathbf{p}, \mathbf{r})} d\mathbf{p}$ while the

nonlinear terms differ: the orders of multiplicity of perturbation in Eq. (6) represent the convolution integrals while in Eq. (15) the corresponding terms have the form

$$\int_{-\infty}^{\infty} W(\mathbf{p}) \hat{q}(\mathbf{p}) e^{-i(\mathbf{p}, \mathbf{r})} \left(\frac{1}{(2\pi)^2} \int_{-\infty}^{\infty} \hat{q}(\mathbf{p} - \mathbf{p}') H(\mathbf{p}') d\mathbf{p}' \right)^k d\mathbf{p}.$$

The inequality of the right sides of Eqs. (6) and (15) may also be demonstrated by the direct test. It is sufficient to consider a variation of the albedo in the form of a simple harmonic

$$\tilde{q}(\mathbf{r}) = \Delta q \cdot \cos(\omega, \mathbf{r}), \tag{16}$$

since any function $\tilde{q}(\mathbf{r})$ can be presented in the form of Fourier integral. Substituting Eq. (16) into Eq. (13) and using the equalities $W(-\omega) = W^*(\omega)$, $C(-\omega) = C^*(\omega)$, and $H(-\omega) = H^*(\omega)$, where * denotes the complex conjugation

$$\text{and } \tilde{q}(\mathbf{p}) = \Delta q \int_{-\infty}^{\infty} \cos(\omega, \mathbf{r}) e^{i(\mathbf{p}, \mathbf{r})} d\mathbf{p} = \Delta q [\delta(\mathbf{p} - \omega) + \delta(\mathbf{p} + \omega)],$$

we obtain

$$\begin{aligned} \tilde{I} &= \frac{\Delta q \bar{E}}{2} \left\{ \frac{W(-\omega) e^{i(x, r)}}{1 - 0.5 \Delta q [H(-2\omega) + H(0)]} + \right. \\ &+ \left. \frac{W(\omega) e^{-i(\omega, r)}}{1 - 0.5 \Delta q [H(0) + H(2\omega)]} \right\} = \\ &= \Delta q \bar{E} \operatorname{Re} \left\{ \frac{W(\omega) e^{-i(\omega, r)}}{1 - 0.5 \Delta q [H(0) + H(2\omega)]} \right\}. \end{aligned} \tag{17}$$

It follows from Eq. (17) that \tilde{I} oscillates with the frequency ω . This result contradicts the physical meaning of the problem: the solution of the direct problem should be nonlinear in $\tilde{q}(\mathbf{r})$ (see formula (6)) due to the photons repeatedly reflected from the underlying surface, and must contain higher harmonics $\cos(n\omega, \mathbf{r})$, $\sin(n\omega, \mathbf{r})$. We can make sure of the latter statement by substituting Eq. (16) into Eq. (6)

$$\begin{aligned} \tilde{I} &= \Delta q \bar{E} \operatorname{Re} \left\{ W(\omega) e^{-i(\omega, r)} + \right. \\ &+ \left(\frac{\Delta q}{2} \right) H(\omega) [W(2\omega) e^{-2i(\omega, r)} + W(0)] + \\ &+ \left(\frac{\Delta q}{2} \right)^2 [W(3\omega) H(-2\omega) H(-\omega) e^{-3i(\omega, r)} + \\ &+ W(\omega) H(-\omega) [H(-2\omega) + H(0)] e^{-i(\omega, r)} + \end{aligned}$$

$$\left. + W(-\omega) H(-\omega) H(0) e^{i(\omega, r)} \right] + O\left(\frac{\Delta q}{2}\right)^3 \}. \tag{18}$$

As expected, the right sides of Eq. (6) and Eq. (13) are identical as $|\omega| \rightarrow 0$:

$$\tilde{I} |_{|\omega| \rightarrow 0} = \Delta q \bar{E} W(0) [1 - \Delta q H(0)]^{-1}.$$

As $|\omega| \rightarrow \infty$ ($z = 0$), we obtain from Eqs. (18) and (17), respectively

$$\tilde{I}_n = (\tilde{I} - \tilde{I}_1) |_{|\omega| \rightarrow \infty} = 0$$

and

$$\tilde{I}_n = (\tilde{I} - \tilde{I}_1) |_{|\omega| \rightarrow \infty} = \frac{0.5 \Delta q H(0)}{1 - 0.5 \Delta q H(0)} \Delta q \bar{E} e^{-v_0/|v| - i(\omega, r)}.$$

Hence, it follows that the relative error in calculating \tilde{I}_n from

$$\text{formula (13)} \quad \gamma = \frac{\tilde{I}_n |_{(18)} - \tilde{I}_n |_{(17)}}{\tilde{I}_n |_{(18)}}$$

increase of ω .

Similar conclusions are true if we take account of the polarization state. Further inaccurate constructions entail formulas (13) and (14) when run down Refs. 7-14. For example, a familiar relation enters in the denominator of the basic expression for the Stokes vector in Ref. 14 (formula (8), p. 405)

$$t(\mathbf{p}) \equiv 1 - \frac{1}{(2\pi)^2} \int_{-\infty}^{\infty} \hat{q}(\mathbf{p}') H(\mathbf{p} - \mathbf{p}') d\mathbf{p}'.$$

Note in conclusion that an account of the horizontal nonuniformity of the albedo leads to the change in the average brightness \bar{I} . The constant term in Eq. (18) indicates to this result

$$\begin{aligned} \Delta q \bar{E} \operatorname{Re} \left[\frac{\Delta q}{2} H(\omega) H(0) + \right. \\ \left. + \left(\frac{\Delta q}{2} \right)^2 W(-\omega) H(-\omega) H(0) + O\left(\frac{\Delta q}{2}\right)^3 \right]. \end{aligned}$$

This term may be neglected in practical calculations.

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