# ON PROBABILITY APPROACH TO MODELS OF ECOLOGICAL REGIONING AND RATIONAL NATURE EXPLOITATION

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*The more general approach to stochastic modeling of pollutant distribution is considered. The model is based on the direct (second) Kolmogorov equation for transitional probabilities written in phase coordinates. Successive closings of this equation give the possibility to construct the distribution laws in the form of probability density function for pollutant concentration.*

Any phenomena (for example, meteorological, hydrological) in the process of development in time include regular and random components. The random component plays a part of interference, which influence on results of economic activity should be minimized.[1] However, to do this, information about not only the value of this interference, but also about the probability of occurrence is necessary. Difficulties of solution to the above problem is often connected with the entirely absent, incomplete, or not representative measurement data. Therefore, when forecasting possible consequences for economic results, very efficient is the mathematical simulation allowing us to try a great variety of situations.

Difficulties connected with the non-ergodicity of natural phenomena (inhomogeneity of processes in time) can be overcome by averaging by cases, rather than in time. As such cases we can take the values of hydrometeorological parameters related to standard periods of observation. Because cases are dated to different years, they can be considered statistically independent with a fair degree of assurance. It should be noted that observations at hydrometeorological stations are discrete set of natural system states. At every instant of time, the system is in one of these states and transits from one state to another with time. Series of such random processes can be considered as a Markovian process without consequences (Markov chain).

Probability density of transition for the Markov chain fits the Smolukhovskii integral equation,[2,3] which solution at certain assumptions about the probabilities of transition leads to solution of the second (direct) Kolmogorov equation

$$\frac{\partial p}{\partial t} + \frac{\partial [A(t, x)p]}{\partial x} = \frac{\partial^2 [B(t, x)p]}{\partial x^2}, \qquad (1)$$

where $p = p(t_0, x_0; t, x)$ is the probability density of system transition from the state $x_0$ to the state $x$ for the time from $t_0$ to $t$. It is natural that over the entirely variation range of $\omega$

$$\int_{\omega} p(t_0, x_0; t, x)\, dx = 1, \quad \text{where } p \geq 0.$$

In one-dimensional case, when the state of system is determined by a single parameter $x$, the coefficient $A(t, x)$ is the mean rate of systematic change of the coordinate $x$; the coefficient $B(t, x)$ is intensity of random oscillations about this mean.

In this paper, construction of probability models is considered using substance spread in liquid and gaseous media as an example.

Let us pass to the phase coordinate $s$ in Eq. (1), where $s$ is pollutant concentration

$$\frac{\partial p}{\partial t} = -\frac{\partial}{\partial s}[A(t, s)p] + \frac{\partial^2}{\partial s^2}[B(t, s)p], \qquad (2)$$

where $p = p(t, s)$; $A = \frac{\partial \bar{s}}{\partial t}$ is the rate of average concentration change in the range $t \in [0, T]$;

$$B = \frac{1}{2}\overline{\left(\frac{\partial s}{\partial t} - \frac{\partial \bar{s}}{\partial t}\right)^2} = \frac{1}{2}\overline{\left(\frac{\partial s'}{\partial t}\right)^2}.$$

Here, the bar above parameters is for averaging, and primed parameters are deviations of instant values for the average ones.

Let us consider the possibility to close Eq. (2). To this end, the equation of transfer and turbulent diffusion of pollutant $s$ is written in divergent form

$$\frac{\partial s}{\partial t} + \frac{\partial u_j s}{\partial x_j} + \alpha s = Q + \nu_{jk}\frac{\partial^2 s}{\partial x_j \partial x_k}, \qquad (3)$$

where $j, k = \overline{1, 3}$ are running numbers of coordinates; $t$ is time; $u_j$ is velocity component along the

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corresponding coordinate $x_j$; $\alpha$ is the coefficient of pollutant non-conservaticity; $Q = Q(x_j, t)$ is the function describing sources of the pollutant under consideration; $\nu_{jk}$ are the turbulent diffusion coefficients. Equation (3) is written in the tensor form, therefore summing is performed by repeating indexes.

Let us average Eq. (3) using the relations $u_j = \overline{u}_j + u_j'$; $s = \overline{s} + s'$; $Q = \overline{Q} + Q'$ and averaging properties.

Let $\alpha$ = const for simplicity,

$$\frac{\partial \overline{s}}{\partial t} + \frac{\partial \overline{u}_j\, \overline{s}}{\partial x_j} + \alpha \overline{s} = \overline{Q} + \nu_{jk} \frac{\partial^2 \overline{s}}{\partial x_j\, \partial x_k} - \frac{\partial\, \overline{u_j'\, s'}}{\partial x_j}\,. \tag{4}$$

Subtracting Eq. (4) from Eq. (3), we have

$$\frac{\partial s'}{\partial t} + \frac{\partial}{\partial x_j}(u_j\, s - \overline{u}_j\, \overline{s}) + \alpha s' = Q' + \nu_{jk} \frac{\partial^2 s'}{\partial x_j\, \partial x_k} + \frac{\partial\, \overline{u_j'\, s'}}{\partial x_j}\,. \tag{5}$$

Let us transform the equation

$$u_j\, s - \overline{u}_j\, \overline{s} = (\overline{u}_j + u_j')(\overline{s} + s') - \overline{u}_j\, \overline{s} = u_j\, s' + \overline{s}\, u_j'$$

and substitute it into Eq. (5):

$$\frac{\partial s'}{\partial t} = -u_j \frac{\partial s'}{\partial x_j} - u_j' \frac{\partial \overline{s}}{\partial x_j} - \left( s' \frac{\partial u_j}{\partial x_j} - \overline{s}\, \frac{\partial u_j'}{\partial x_j} \right) -$$

$$- \alpha s' + Q' + \nu_{jk} \frac{\partial^2 s'}{\partial x_j\, \partial x_k} + \frac{\partial\, \overline{u_j'\, s'}}{\partial x_j}\,. \tag{6}$$

In Eq. (6) every term in parenthesis is divergence correspondingly for velocity and its fluctuation. For incompressible liquid these terms vanish.

Following Refs. 4 and 5, in linear approximation the influence of pollutant on the medium velocity field can be neglected, i.e. the turbulent field of medium velocity can be considered independent of pollutant concentration. Let us introduce the designation

$$q_i = \overline{u_i'\, s'}\,. \tag{7}$$

Here $s'$ is the unknown parameter, $i = \overline{1, 3}$.

Let us perform averaging in time $T \gg \tau$ ($\tau$ is the Euler scale). Correcting the detected mistakes from Refs. 4 and 5, we integrate Eq. (6) with respect to time from $t$ to $t + \tau$

$$s'(t+\tau) = s'(t) + \int_t^{t+\tau} \left[ - u_j \frac{\partial s'}{\partial x_j} - u_j' \frac{\partial \overline{s}}{\partial x_j} - \right.$$

$$\left. - \alpha s' + Q' + \nu_{jk} \frac{\partial^2 s'}{\partial x_j\, \partial x_k} + \frac{\partial\, \overline{u_j'\, s'}}{\partial x_j} \right] \mathrm{d}t.$$

To perform the averaging (7), we multiply both part of the latter equation by $u_i'(t+\tau)$ and average it in the range $T - \tau$

$$q_i = \frac{1}{T-\tau} \int_0^{T-\tau} u_i'(t+\tau)\, s'(t+\tau)\, \mathrm{d}t =$$

$$= \frac{1}{T-\tau} \int_0^{T-\tau} u_i'(t+\tau)\, s'(t)\, \mathrm{d}t - \frac{1}{T-\tau} \int_0^{T-\tau} u_i'(t+\tau) \times$$

$$\times \int_t^{t+\tau} u_j(t_1) \frac{\partial s'}{\partial x_j}\, \mathrm{d}t_1\, \mathrm{d}t - \frac{1}{T-\tau} \int_0^{T-\tau} u_i'(t+\tau) \times$$

$$\times \int_t^{t+\tau} u_j'(t_1) \frac{\partial \overline{s}}{\partial x_j}\, \mathrm{d}t_1\, \mathrm{d}t - \frac{\alpha}{T-\tau} \int_0^{T-\tau} u_i'(t+\tau) \times$$

$$\times \int_t^{t+\tau} s'\mathrm{d}t_1\, \mathrm{d}t + \frac{\nu_{ik}}{T-\tau} \int_0^{T-\tau} u_i'(t+\tau) \times$$

$$\times \int_t^{t+\tau} \frac{\partial^2 s'}{\partial x_j\, \partial x_k}\, \mathrm{d}t_1\, \mathrm{d}t + \frac{1}{T-\tau} \int_0^{T-\tau} u_i'(t+\tau) \times$$

$$\times \int_t^{t+\tau} Q'\mathrm{d}t_1\, \mathrm{d}t + \frac{1}{T-\tau} \int_0^{T-\tau} u_i'(t+\tau) \int_t^{t+\tau} \frac{\partial\, \overline{u_j'\, s'}}{\partial x_j}\, \mathrm{d}t_1\, \mathrm{d}t\,. \tag{8}$$

The first term in the right-hand side of Eq. (8) vanishes due to non-correlation of the functions $u_i'(t+\tau)$ and $s'(t)$ under the integral sign, the last term vanishes in virtue of $T \gg \tau$.

From Eq. (8), according to Refs. 4 and 5, we can obtain the first approximation for Eq. (7)

$$\overline{u_i'\, s'^{(1)}} = -K_{ij}^{(1)} \frac{\partial \overline{s}}{\partial x_j} + Q'^{(1)}_i\,, \tag{9}$$

where

$$K_{ij}^{(1)} = \frac{1}{T-\tau} \int_0^{T-\tau} u_i'(t+\tau) \int_t^{t+\tau} u_j'(t_1) \frac{\partial \overline{s}}{\partial x_j}\, \mathrm{d}t_1\, \mathrm{d}t;$$

$$Q_i^{(1)} = \frac{1}{T-\tau} \int_0^{T-\tau} u_i'(t+\tau) \int_t^{t+\tau} Q' \mathrm{d}t_1 \, \mathrm{d}t.$$

Having substituted Eq. (9) into Eq. (4), we close the equation for $\bar{S}$:

$$\frac{\partial \bar{s}}{\partial t} = -\frac{\partial \bar{u}_j \bar{s}}{\partial x_j} - \alpha \bar{s} + \bar{Q} +$$

$$+ \nu_{jk} \frac{\partial^2 \bar{s}}{\partial x_j \partial x_k} + \frac{\partial}{\partial x_j} K_{ij}^{(1)} \frac{\partial \bar{s}}{\partial x_j} - Q'^{(1)}_i \,, \qquad (10)$$

the left-hand side of which is designated by the coefficient $A$ in Eq. (2).

Initial conditions for Eq. (10) are as follows: $\bar{s} = s_0$ at $t = 0$. The boundary conditions at horizontal boundaries of the integration range $D\{-X \le x \le X, -Y \le y \le Y\}$ and at the top boundary at $z = Z$ are set in the following way. At that points of boundaries, where the velocity vector is directed inward the area of solution determination, $\bar{s} = s_\Phi$. At the points, where the velocity vector is directed outward this area, values of concentration are extrapolated to the boundary using near-boundary values with the second order of approximation. At the low boundary at $z = \Delta$, the boundary condition of the third order is set, which takes into account pollutant absorption and reflection. Here $s_0$ and $s_\Phi$ are given values. Equation (10) can be solved by numerical integration in the Cartesian rectangular coordinate system with use of the method of dummy areas.[6] Finite-difference approximations of derivatives in spatial variables are constructed based on the integro-interpolation method.[6] Problem approximation with respect to time is constructed using dicyclic complete splitting. The used scheme of splitting by components gives the solution for non-commutative operators with the second order of approximation in time and coordinates. For numerical realization of finite-difference equations, the non-monotonic run is used.

Solution of Eq. (10) is also of independent interest, because it allows calculation of both the field of average concentrations and pollutant accumulation on the underlying surface.

To determine the coefficient $B$ in Eq. (2), we use Eq. (6) substituting the equation of recursive nest (9) into it. Then Eq. (6) should be squared and averaged. Having determined the coefficients $A$ and $B$, let us integrate Eq. (2) using the same numerical method as for Eq. (10). The obtained solution allows description of pollutant concentration behavior from the purely probability point of view.

The above ideas were realized in different particular problems solved by the author.[7]

In spite of the concentration $s$, other characteristics of the medium can be considered, for example, humidity or temperature. In the general case, the probability density function in the Kolmogorov equation can be considered as the function of several variables.

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