METHODS FOR SOLVING THE PHASE PROBLEM IN DIGITAL PROCESSING OF IMAGE. PART I. THEORETICAL ASPECTS OF THE PHASE PROBLEM

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The phase problem and algorithms of its solution are analyzed theoretically. The questions on the ambiguity of the phase problem solution in the one–dimensional and multi–dimensional cases are discussed. A new method is proposed for constructing all solutions in a two–dimensional case.

In many fields of applied optics it often happens so that the only accessible and undistorted information about the sought–after spatially–limited (finite) distribution is the modules of its Fourier spectrum. Such a situation occurs, for example, in astronomy when the Labeyrie methods and the intensity holograms are used for processing of the object images distorted by the turbulent atmosphere.

Starting from the end of the 1950–th great attention was paid to the solution of this problem named as the phase problem. Both universal theoretical results concerning the form and number of solutions[1–10] and particular schemes for reconstruction[11–19] have been obtained by recent time. However, a detailed theoretical analysis of a two–dimensional case of the phase problem, in particular, for discrete distributions has not yet been completed. The question on developing a fast and stable algorithm of reconstruction has not yet been resolved either. In this paper these questions are considered based on already known and original results in application to a very practical problem on digital processing of optical images.

MATHEMATICAL FORMULATION OF THE PROBLEM

Let $J(\mathbf{t})$ be a continuous positive function (distribution over an image) which is nonzero in a finite region of the space S. Let us define its Fourier transform as follows:

$$f(\mathbf{x}) = \hat{F}\{J\} = \int\limits_S J(\mathbf{t}) \exp\{i\mathbf{x}\mathbf{t}\} \,\mathrm{d}t = A(\mathbf{x}) \exp\{i\varphi(\mathbf{x})\} , \qquad (1)$$

where $\hat{F}$ is the Fourier transform operator; $A(\mathbf{x})$ is the known modulus of spectrum; $\varphi(\mathbf{x}) = \arg f(\mathbf{x})$ is an unknown phase of the spectrum.

In the case of inverse Fourier transform of $A^2(\mathbf{x})$ we obtain an autocorrelation equation for the image

$$Q(\mathbf{t}) = \int\limits_S J(\mathbf{t}_1)\, J(\mathbf{t}_1 + \mathbf{t})\, \mathrm{d}\mathbf{t}_1 , \qquad (2)$$

where

$$Q(\mathbf{t}) = F^{-1}\{A^2(\mathbf{x})\} .$$

By applying a generalized Fourier transform defined for the complex values of the variables $\mathbf{w} = \mathbf{x} + i\mathbf{y}$ to Eq. (2) one obtains one more equation of the phase problem

$$f_Q(\mathbf{w}) = f(\mathbf{w})\, f(-\mathbf{w}) . \qquad (3)$$

For digital processing a distribution of signal over image is described by a set of positive readings of the form $\{J(n_1, n_2): 0 \le n_1 \le N_1,\ 0 \le n_2 \le N_2\}$, and the autocorrelation Q can be given in the form

$$Q(m_1, m_2) = \sum_{n_1=0}^{N_1} \sum_{n_2=0}^{N_2} J(n_1, n_2)\, J(n_1 + m_1, n_2 + m_2) , \qquad (4)$$

while the expression for $f(\mathbf{w})$ takes the form

$$f(w_1, w_2) = \sum_{n_1=0}^{N_1} \sum_{n_2=0}^{N_2} J(n_1, n_2) \exp\{iw_1n_1 + iw_2n_2\} .$$

Change of variables $z_1 = \exp\{iw_1\}$, $z_2 = \exp\{iw_2\}$ allows Eq. (3) to be written in the form

$$R_Q(\mathbf{z}) = R_J(\mathbf{z})\, R_J(\mathbf{z}^{-1}) , \qquad (5)$$

where $R_J(\mathbf{z}) = \sum_{n_1=0}^{N_1} \sum_{n_2=0}^{N_2} J(n_1, n_2)\, z_1^{n_1} z_2^{n_2}$ is the z–image which is a two–dimensional polynomial.

Thus, we obtain the following equivalent formulations of the phase problem as of the problem on reconstruction of:

1) the phase $\varphi(\mathbf{x})$ from the modulus $A(\mathbf{x})$ (for a discrete case $\varphi(\mathbf{z}) = \arg R_J(\mathbf{z})$ from $|R_J(\mathbf{z})|$ at $|\mathbf{z}| = 1$),

2) the image $J(\mathbf{t})$ from Eq. (2) (for the discrete case of the image $J(n_1, n_2)$ from Eq. (4)), and

3) the unknown integral analytical function $f(\mathbf{w})$ from Eq. (3) (for a discrete function of the polynomial $R_J(z_1, z_2)$ from Eq. (5).

UNIQUENESS OF THE RECONSTRUCTION

It is well known that an ordinary shift of an image without any change in its structure and shape $J_1(t_1, t_2) = J(t_1 + \alpha_1, t_2 + \alpha_2)$ results only in a change of its Fourier phase by a linear term $\varphi_1(x_1, x_2) = \varphi(x_1, x_2) + \alpha_1x_1 + \alpha_2x_2$ while the Fourier spectrum of the mirror image $J(t_1, t_2) = J(-t_1, -t_2)$ is a complex conjugate to the initial spectrum $\varphi_1(x_1, x_2) = -\varphi(x_1, x_2)$. Thus, it follows from the problem formulation itself, that two Fourier spectra $f(x_1, x_2)$ and $f(\pm x_1, \pm x_2) \exp\{i(\alpha_1x_1 + \alpha_2x_2)\}$ are the equivalent solutions of the phase problem, i.e., the image can be reconstructed with an accuracy up to the shift and turn by 180°.

General analysis of uniqueness of the problem solution can be easily carried out by considering Eq. (3). Here the first factor $f(\mathbf{w})$ corresponds to the real image, while the second factor $f(-\mathbf{w})$ – to the mirror image, therefore, the extraneous solutions (nonequivalent) can appear only in the case when these factors are "mixed". Let the initial image be a convolution of two subimages $J(\mathbf{t}) = J_1(\mathbf{t}) * J_2(\mathbf{t})$. In this case Eq. (3) takes the form $f_Q(\mathbf{w}) = f_1(\mathbf{w})f_2(\mathbf{w})f_1(-\mathbf{w})f_2(-\mathbf{w})$, from which we obtain all solutions $f(\mathbf{w}) = f_1(\mathbf{w})f_2(\mathbf{w})$, $f(\mathbf{w}) = f_1(-\mathbf{w})f_2(-\mathbf{w})$, $f(\mathbf{w}) = f_1(\mathbf{w})f_2(-\mathbf{w})$, $f(\mathbf{w}) = f_2(\mathbf{w})f_1(-\mathbf{w})$. Thus, formally, in this case the problem has four solutions with the quite obvious differences between them, if we represent their Fourier spectra in the form of Eq. (1) (for $y = 0$ and $\mathbf{w} = \mathbf{x}$ on the real axes)

$$f(\mathbf{x}) = A_1 e^{i\varphi_1} A_2 e^{i\varphi_2}, \quad f(-\mathbf{x}) = A_1 e^{-i\varphi_1} A_2 e^{-i\varphi_2},$$

$$f(\mathbf{x}) = A_1 e^{i\varphi_1} A_2 e^{-i\varphi_2}, \quad f(-\mathbf{x}) = A_1 e^{-i\varphi_1} A_2 e^{i\varphi_2}.$$

The spectrum moduli of all solutions are the same and equal to $A_1(\mathbf{x})\cdot A_2(\mathbf{x})$. The first two solutions are equivalent true solutions, while the rest two solutions are equivalent but extraneous solutions, i.e., in fact, there are only two solutions: $J_1(\mathbf{t}) * J_2(\mathbf{t})$ and $J_1(\mathbf{t}) * J_2(-\mathbf{t})$. If the initial image is a convolution N of subimages $J(\mathbf{t}) = J_1(\mathbf{t}) *\ldots* J_N(\mathbf{t})$ then Eq. (3) has the form

$$f_Q(\mathbf{w}) = f_1(\mathbf{w})\, f_2(\mathbf{w}) \ldots f_N(\mathbf{w})\cdot f_1(-\mathbf{w})\, f_2(-\mathbf{w}) \cdots f_N(-\mathbf{w}).$$

New solutions having the same modulus on the real axes ($\mathbf{w} = \mathbf{x}$) but different phases as $\prod_{i=1}^{N} f_i(\pm\mathbf{w})$ can be easily derived from Eq. (6). The total number of such variants is equal to 2^{N-1}. In this case all of them possess one and the same autocorrelation. Since linear size of the autocorrelation region two times exceeds the size of image region S for positive definite images, then all relations satisfying the positiveness condition are solutions of the phase problem.

For a discrete case the analysis is absolutely analogous and is based on relation (5), which in this case takes the following form

$$R_Q(\mathbf{z}) = R_1(\mathbf{z})\, R_2(\mathbf{z}) \ldots R_N(\mathbf{z})\cdot R_1(\mathbf{z}^{-1})\, R_2(\mathbf{z}^{-1}) \ldots R_N(\mathbf{z}^{-1}),$$

and new solutions of the form $\prod_{i=1}^{N} R_i(\mathbf{z})$ can be obtained by replacing any of terms $R_i(\mathbf{z})$ by $R_i(\mathbf{z}^{-1})$.

Note that in the particular case of a symmetric image $J(\mathbf{t}) = J(-\mathbf{t})$ the problem can be solved unequivocally since $f(-\mathbf{w}) = f(\mathbf{w})$ and $R_J = R_J(\mathbf{z})$. Therefore, when deriving new solutions we can use only the factors corresponding to nonsymmetric subimages for which $J_i(-\mathbf{t}) \neq J_i(\mathbf{t})$, $f_i(-\mathbf{w}) \neq f_i(\mathbf{w})$, and $R_i(\mathbf{z}^{-1}) \neq R_i(\mathbf{z})$.

As follows from the above the general question on unambiguity of the phase problem is reduced to the question on representability of an unknown image in the form of a convolution of subimages or on the factorability of a Fourier spectrum, with each of the factors being in correspondence with a finite positive function.

1. One – dimensional case. The function $f(w)$, as an integral exponential function, can be represented in the form of a canonical Adamar–Weierstrass[21] product

$$f(w) = \exp(\beta_0 + \beta_1 w) \prod_{i}^{\infty} \left(1 - \frac{w}{w_i}\right) \exp\left(\frac{w}{w_i}\right),$$

where w_i are the roots of equation $f(w) = 0$ and β_0 and β_1 are the complex constants. A new solution can be obtained by changing the root w_i in the expansion $f(w)$ for w_i^*, in so doing, the number of new solutions is unlimited, in principle, but it should be noted that only those roots can give a new solution which are not on the real axis, and provided that the images corresponding to them satisfy the condition of positiveness.

In a discrete case the z–image $R_J(z)$ is the one–dimensional polynomial of Nth power, it has N roots and can be represented in the form

$$R_J(\mathbf{z}) = C \prod_{i=1}^{N} (z - z_i),$$

where z_i are the roots of equation $R_J(z) = 0$ and C is the complex constant.

A new solution for the case of real roots $z_i = z_i^*$ can be obtained by replacing z_i for z_i^{-1} in any of the factors.[3] When $z_i \neq z_i^*$ the roots have to be replaced by pairs because a new solution consists of the factors $(z - z_i)\left(z - \frac{1}{z_i^*}\right)$, where z_i is an arbitrary root of the pair (z_i, z_i^*). Here we can obtain $\sim 2^{N_1+N_2}$ solutions, where N_1 is the number of real roots, and N_2 is the number of conjugate pairs of roots which do not belong to the circle $|z| = 1$.

2. Two–dimensional case. In accordance with Ref. 22 there exists a generalized multi–dimensional analog of the canonical product in the form

$$f(\mathbf{w}) = H(\mathbf{w})\, G(\mathbf{w}).$$

However, specific views of $H(\mathbf{w})$ and $G(\mathbf{w})$ are unknown and, moreover, there are no proofs that finite positive functions will correspond to them in the region where the image is defined. At this point let us stop the consideration of the continuous case for a while.

In the discrete case the question on the solution unambiguity is reduced to the question on the possibility of factorizing two–dimensional polynomial $R_J(z_1, z_2) = \sum_{n_1=0}^{N_1} \sum_{n_2=0}^{N_2} J_{n_1 n_2} z_1^{n_1} z_2^{n_2}$ into the product of polynomials of lower power. In the general case such a factorisation is impossible. To estimate the probability of obtaining an unambiguous solution of the phase problem it is convenient to use the method of the Lebesgue measure. The analysis is based on two statements.[8]

Statement 1. Let the correspondence of $\hat{Q}$ points of an m–dimensional set B^m to the points of an n–dimensional set $B^n(\hat{Q}: B^m \to B^n)$ be continuously differentiable and $m<n$. Then the Lebesgue measure of a subset which is a $\hat{Q}$–image $B^m(\hat{Q}[B^m])$ in the set B^n is equal to zero.

In other words, in an n–dimensional space the Lebesgue measure of any subset determined by the

m–independent parameters (an optional set) is equal to zero at $m<n$ (say, the area of the curve dependent only of a single parameter is equal to zero).

Denote the set of polynomials of the power n with k variables as $P(n, k)$. An individual polynomial $P_n(\mathbf{z})$ from this set can be represented in the form

$$P_n(\mathbf{z}) = \sum_{l_1+l_2+\ldots+l_k \le n} C(l_1 \ldots l_k)\, z_1^{l_1} z_2^{l_2} \ldots z_k^{l_k} .$$

The number of coefficients determining the polynomial from $P(n, k)$ will be denoted as $\alpha(n, k)$, and the space of coefficients of the polynomials as $R^{\alpha(n, k)}$. Since each coefficient of the polynomial $P_n(\boldsymbol{z})$ can be represented as a coordinate of a vector from $R^{\alpha(n, k)}$ then there exists one-to-one correspondence between $P(n, k)$ and $R^{\alpha(n, k)}$.

Statement 2. The subset B of factorable polynomials of the set $P(n, k)$ corresponds to the set with a zero measure in $R^{\alpha(n, k)}$ under conditions that $k>1$ and $n>1$.

Based on these statements one can easily come to several important conclusions. For example, in the one–dimensional case Eq. (5) can be unambiguously solved either for symmetrical images or for the images which are a convolution of a symmetrical subimage and a nonfactorable–more symmetric subimage. Therefore, only $\frac{N+2}{2}$ from N parameters describing the image of the length $(N + 1)$ can be independent. At the same time the set of autocorrelations $\{Q_n;\ -N \le n \le N\}$ contains N independent parameters. Consequently, the Lebesgue measure of a subset of autocorrelations which can be unequivocally solved in the set of all autocorrelations is equal to zero for $N \ge 3$ while the phase problem for $N \ge 3$ is almost always resolved ambiguously.

In a two–dimensional case the solution is unique provided that the z–image expansion involves only one asymmetrical factor. As follows from Statement 2 the mathematical probability of occurrence of factorable two–dimensional and multi–dimensional polynomials is equal to zero, hence, as a rule, the phase problem can be solved in the two–dimensional and multidimensional cases unambiguously. The words "as a rule" and "almost always" mean here, that the given statements are valid for each element of the set excluding for a subset with the zero Lebesgue measure.

The above–mentioned discussions make it possible to understand the difference between the one–dimensional and multi–dimensional cases and explain the success of numerical calculations carried out for two–dimensional case of the problem. However, there appears a question on the method of obtaining all possible solutions in the two–dimensional case in which an initial image is in fact a convolution of two– and one–dimensional subimages. If in the one–dimensional case this question can be answered by finding all zeros of Eq. (5) and constructing all solutions by the above–described transfer of the roots, in the two–dimensional case it is impossible to factorize the polynomial directly. Therefore, the authors propose a general method of reducing a two–dimensional discrete case to a one–dimensional case. Its basic idea is in a row–by–row elongation of the discrete image J_{n_1, n_2}, i.e., a one–dimensional image is put in correspondence to it, according to the rule

$$I_n = J_{n_1, n_2} \text{ at } n = n_1 + n_2(N_1 + 1) .$$

Let us establish the correspondence between the values of autocorrelation Q_l of the image I_n and autocorrelation Q_{l_1, l_2} of the image J_{n_1, n_2}. Since z–images of both autocorrelations satisfy Eq. (5) and equality $R_I(z) = R_J(z, z^{N_1+1})$ is true for z–images (with an account of Eq. (7)), we can write down

$$R_Q(z) = R_Q\left(z, z^{N_1+1}\right) .$$

In accordance with Eq. (5) and with an account of the view of the polynomial, $R_Q(z, z^{N_1+1})$ can be represented as

$$R_Q\left(z, z^{N_1+1}\right) = \left\{ \sum_{n_1=0}^{N_1} \sum_{n_2=0}^{N_2} J_{n_1, n_2} z^{n_1 + n_2(N_1+1)} \right\} \times$$

$$\times \left\{ \sum_{m_1=0}^{M_1} \sum_{m_2=0}^{M_2} J_{m_1, m_2} z^{-m_1 - m_2(N_1+1)} \right\} .$$

Writing down an analogous expression for $R_Q(z)$ with an account of Eq. (8) we can find

$$Q_l = \sum_{n_1=0}^{N_1} \sum_{n_2=0}^{N_2} \sum_{m_1=0}^{M_1} \sum_{m_2=0}^{M_2} J_{n_1, n_2} J_{m_1, m_2}\, \delta\{l - (n_1 - m_1) -$$

$$- (n_2 - m_2)(N_1 + 1)\} .$$

Subsequent analysis is then reduced to finding the conditions under which an argument of the δ–function becomes zero, that can be given in the form of a system of equations

$$\begin{cases} n_1 - m_1 = l_1 - p(N_1 + 1) , \\ n_2 - m_2 = l_2 + p , \end{cases} \tag{9}$$

where $l_2 = \left[\dfrac{l}{N_1 + 1}\right]$, $l_1 = l - \left[\dfrac{l}{N_1 + 1}\right](N_1 + 1)$, and p is the discrete parameter determining the range of series expansion over the modulus of $(N_1 + 1)$. From Eq. (9) we can derive a system of inequalities determining the range of values p: $0 \le l_1 \le N_1$, $|l_1 - p(N_1 + 1)| \le N_1$. The graphic representation of the inequalities shows that p can take only two values: 0 and or 1.

Therefore, the value of the one–dimensional autocorrelation Q_l of the image I_n elongated in row is the sum of two autocorrelations Q_{l_1, l_2} (see Ref. 20):

$$Q_l = Q_{l_1, l_2} + Q_{l_1 - N_1 - 1,\, l_2 + 1} , \tag{10}$$

where

$$l_1 = l - \left[\frac{l}{N_1 + 1}\right](N_1 + 1) ,\ l_2 = \left[\frac{l}{N_1 + 1}\right] .$$

A column–by–column elongation, in accordance with the rule $I_n = J_{n_1, n_2}$ and $n = n_2 + n_1(N_2 + 1)$ results in

$$Q_l = Q_{l_1, l_2} + Q_{l_1+1,\, l_2 - N_2 - 1} , \tag{11}$$

where

$$l_1 = \left[\frac{l}{N_2 + 1}\right], \; l_2 = l - \left[\frac{l}{N_2 + 1}\right](N_2 + 1) .$$

Let us consider a row–by–row method of elongation with zeros which is reduced to adding a row consisting of N_1 zeros to the last element of each row of a two–dimensional image.[10] As a result, only one component remains in Eqs. (10) and (11). The rule of elongation is given in the form

$$I_n = J_{n_1,n_2} , \quad n = n_1 + n_2(2N_1 + 1) , \tag{12}$$

and Eq. (8) takes the form

$$R_Q(z) = R_Q\left(z, z^{2N_1+1}\right) . \tag{13}$$

As in the preceeding case we obtain a system of equations

$$\begin{cases} n_1 - m_1 = l_1 - p(2N_1 + 1) , \\ n_2 - m_2 = l_2 + p , \end{cases}$$

where $l_2 = \left[\frac{l}{2N_1 + 1}\right], \; l_1 = l - \left[\frac{l}{2N_1 + 1}\right](2N_1 + 1)$.

The system of inequalities from which we find the range of values of the parameter p is $0 \le l_1 \le 2N_1$ and $|l_1 - p(2N_1 + 1)| \le N_1$.

The graphical representation of these inequalities shows that p can take only one value: either 0 or 1. Therefore Eq. (10) takes the form

$$Q_l = Q_{l_1,l_2} . \tag{14}$$

In this case $Q_{l_1-2N_1-2,\, l_2+1} = 0$, or

$$Q_l = Q_{l_1-2N_1-2,\, l_2+1} \tag{15}$$

where

$$Q_{l_1,\, l_2} = 0 , \; l_1 = l - \left[\frac{l}{2N_1 + 1}\right](2N_1 + 1) , \; l_2 = \left[\frac{l}{2N_1 + 1}\right] .$$

It is obvious that Eqs. (14) and (15) are valid in the case of column–by–column elongation with zeros as well.

In the general case the row–by–row elongation of $J_{n_1,\, n_2}$ into a one–dimensional image can be given in the form $I_n = J_{n_1,\, n_2}$ and $n = n_1 + n_2M_1$, that results in:

a) $M_1 = N_1 + 1$ elongation without zeros (10),

b) $N_1 + 1 < M_1 < 2N_1 + 1$ is an intermediate case in which the elongation is done with zeros but the number of zeros is insufficient to ensure an equality to zero of one of the terms of Eq. (10) and Q will also be a sum of two two–dimensional autocorrelations, one of them being shifted,

c) $M_1 = 2N_1 + 1$, we obtain either Eq. (14) or Eq. (15),

d) $M_1 > 2N_1 + 1$, the method is applicable but it is rather difficult to establish the correspondence of the form (14) or (15), since it cannot be written analytically, and

e) $1 < M_1 < N_1 + 1$, in this case Q_l is a combination of three and more two–dimensional autocorrelations but similar elongation is now senseless, because the rows are superposed and it becomes impossible to reconstruct the image $J_{n_1,\, n_2}$ from I_n.

The method developed here makes it possible to formulate an algorithm for finding all solutions of the phase problem in a two–dimensional discrete case.

1. From the given autocorrelation $Q_{l_1,\, l_2}$ within the region $S_Q(-L_1 \le l_1 \le L, -L_2 \le l_2 \le L_2)$ the region S in which the image is defined as $N_1 = [L_1] + 1$ and $N_2 = [L_2] + 1$.

2. In accordance with N_1 and N_2 we formulate the rule of an elongation of an unknown two–dimensional image $J_{n_1,\, n_2}$ into a one–dimensional I_n image either row–by–row $n = n_1 + n_2M_1(N_1 + 1 \le M_1 \le 2N_1 + 1)$ or column–by–column $n = n_2 + n_1M_2\ (N_2 + 1 \le M_2 \le 2N_2 + 1)$.

3. Depending on the chosen rule of elongation the one–dimensional autocorrelation Q_l of a one–dimensional analog of the image I_n is constructed.

4. The roots $R_Q(z)$ and all solutions of a one–dimensional discrete case of the problem are found.[3]

5. Thusly obtained one–dimensional solutions are then selected out according to the criterion:

a) contractibility into a two–dimensional image; if the elongation occurred with zeros, then they have to possess zero values at the corresponding regions, and

b) if the elongation was without zeros, then their one–dimensional autocorelation Q_l is equal to the sum of two two–dimensional autocorrelations, therefore, it is necessary to calculate one–dimensional autocorrelations of the obtained solutions Q_l^m, to calculate convolutions into the two–dimensional ones $\rightarrow O^m_{l_1,\, l_2}$, and to check an equality $Q_{l_1,\, l_2} = O^m_{l_1,\, l_2}$. In this case the extraneous solutions do not satisfy this equality.

It is possible to prove theoretically, that the Lebesgue measure of the constructed extraneous solutions under the above–mentioned restrictions is equal to zero.

Further discussions of this problem will be continued in Parts 2 and 3 of this paper.

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