METHOD OF MULTIPLE REFLECTIONS IN THEORY OF RADIATIVE TRANSFER. SOME CRITICAL COMMENTS

A.G. Borovoi

Institute of Atmospheric Optics, Siberian Branch of the Russian Academy of Sciences, Tomsk
Received February 10, 1993

The method of multiple reflections is in fact the modification of a two–flux approximation. Accuracy of this method is analyzed. It is pointed out that feasibility of the method is problematic.

In 1978 B.A. Savel'ev reported on a method for solving the radiative transfer equation and called it the method of multiple reflections.[1] By now Savel'ev et al. published more than ten papers concerning this problem (see Refs. 2–4 and the references cited therein) in which the method was used primarily for numerical solution of various problems.

Below a critical review of this method is given.

1. CONCEPT OF THE METHOD

At present theory of radiative transfer is in fact one of the divisions of mathematical physics concerned with a solution of radiative transfer equation. The physical meaning of the radiative transfer equation is very simple: this is a balance equation for the number of radiated particles, which we call the photons for short, in an elementary volume. The simplest form of this equation is as follows:

$$\frac{\mathrm{d}I}{\mathrm{d}l} = -\alpha I + \int \beta(\mathbf{n}' \to \mathbf{n})\, I(\mathbf{r}, \mathbf{n}')\, \mathrm{d}\mathbf{n}' . \qquad (1)$$

It means that the ray intensity $I(\mathbf{r}, \mathbf{n})$, i.e., the density of photons at the point $\mathbf{r}$ having the velocity direction $\mathbf{n}(|\mathbf{n}| = 1)$, diminishes along the ray $l = \mathbf{r} + \xi\mathbf{n}$ $(\xi > 0)$ due to collisions with scatterers according to exponential law with the extinction coefficient α. The coefficients of equation are usually taken in the form

$$\alpha = c\,\sigma , \qquad \beta(\mathbf{n}' \to \mathbf{n}) = c\,\sigma_s\, p(\mathbf{n}' \to \mathbf{n}) , \qquad (2)$$

where c is the number density of the scatterers, σ and σ_s are the cross sections of extinction and scattering by a single scatterer, $p(\mathbf{n}' \to \mathbf{n})$ is the probability density for a photon having the velocity direction $\mathbf{n}'$ to have the direction $\mathbf{n}$ after a collision, the function $p(\mathbf{n}' \to \mathbf{n})$ is referred to as the scattering phase function. The last term of Eq. (1) means that in addition to the process of photon absorption by the scatterers there occurs a partly compensating process of formation of photons moving along the ray l due to their scattering from the other directions $\mathbf{n}'$.

Now let us assume that for some reasons the photons can move only along one direction, for example, along the x axis forward and backward. In this case the ray intensity being considered initially as a function in space of five variables $\mathbf{r}$ and $\mathbf{n}$ degenerates into two functions: the density of photons moving along the x axis $I_1(x)$ and the density of photons moving in the opposite direction $I_2(x)$. Balance equation for the number of radiated particles (1) degenerates into the system of two equations

$$\frac{\mathrm{d}I_1}{\mathrm{d}x} = -\alpha' I_1 + \beta' I_2 ,$$
$$\frac{\mathrm{d}I_2}{\mathrm{d}(-x)} = -\alpha' I_2 + \beta' I_1 , \qquad (3)$$

where the coefficients α' and β' are numerical

$$\beta' = c\,\sigma_s\, p'(\mathbf{n}_0 \to -\mathbf{n}_0) , \qquad \alpha' = \beta'' + c\,\sigma_{\mathrm{ab}} , \qquad (4)$$

$\mathbf{n}_0$ is the unit vector along the x axis, $p'(\mathbf{n}_0 \to -\mathbf{n}_0) + p'(\mathbf{n}_0 \to \mathbf{n}_0) = 1$, and σ_{ab} is the absorption cross section.

The system of equations (3) is referred to as one–dimensional radiative transfer equation. In comparison with general equation (1) in space of five variables whose solution is a matter of great concern of the entire division of mathematical physics, the one–dimensional transfer equation has a great advantage, namely, it can be solved in a simplest way. But this brings up the question of whether there are physical objects in nature which can be described by the one–dimensional transfer equation.

A pile of plane–parallel plates which transforms the perpendicularly incident light into the photons moving along the x axis in forward or backward direction can be seemingly considered as such an object. However, this problem can be solved more rigourously using the Maxwell equations. Interference of the scattered waves proved to be of great importance in this problem. It was shown that even if a stratified randomly inhomogeneous medium is taken into consideration in which interference is expected to be insignificant because of the randomness of the reflected wave phases and the model concept of radiation as an ensemble of particles (photons) seems to be appropriate, the equation for the intensity differs from the one–dimensional radiative transfer equation (see, for example, Ref. 5).

A pile of rough plates or a scattering medium consisting of a great number of small particles analogous to it, for example, clouds would be considered as the second physical object. It is well known that here interference of scattered waves is insignificant. In this case the Maxwell equations for the intensity are reduced to the radiative transfer equation but in its general form (1) in space of five variables, since here each scatterer produces photons moving in all directions.

As to the problems with plane symmetry, for example, in the case of homogeneous layer and radiation incident normally on it, the fluxes of photons

0235-6880/93/05 363-04 $02.00

$$I'_1 = \int\limits_{\mathbf{nn}>0} I(\mathbf{r},\ \mathbf{n})(\mathbf{nn}_0)\mathrm{d}\mathbf{n}\ ,\ I'_2 = \int\limits_{\mathbf{nn}<0} I(\mathbf{r},\ \mathbf{n}(\mathbf{nn}_0)\mathrm{d}\mathbf{n} \qquad (5)$$

by approximate integration of Eq. (1) can be reduced to a system of two equations which is in fact equivalent to system (3). Needless to say that the coefficients α' and β' are no longer determined from relations (4). This system of equations is referred to as a two–flux approximation (see Refs. 6 and 7).

The two–flux approximation was used for practical estimates by the originators of radiative transfer equation (1) even in the 19th century. However, it has not yet been widely used. Usually, one mentions its use in paint and varnish industry to estimate the parameters of particles making up paint.

A key point of the problem of the two–flux approximation is not a solution of the system of equations (3), which is simple, but a determination of the coefficients α' and β' in terms of the parameters of a scattering medium.[6,7]

Let us consider one more physical object which can be described by the one–dimensional transfer equation. This object is a thin long bar or cylinder consisting of the light scattering particles. The radiation is incident on the end of it. Ideally, this is a one–dimensional chain of scatterers strung, for example, on the x axis. The radiation leaving the ends of this cylinder in the direction of the x axis is caused by multiple scattering in which a single scattering event is described by the coefficients in the form of Eq. (4). But it is easy to show that one–dimensional equation (3) is inapplicable in this case because the intensity of radiation scattered on a single scatterer will diminish along the x axis as x^{-2}, whereas the system of equations (3) does not describe such a diminution.

Thus, one–dimensional radiative transfer equation (3) has a physical meaning only as a two–flux approximation of general equation (1).

Let us now proceed to the concept of multiple reflection method based on the physical concepts presented above.

Goryachev et al.[2] pretended to develop a "general heuristic approach" and "semi–analytical" method for calculating the radiation fluxes in the scattering media of arbitrary geometry. However, actually we have a calculation algorithm for a scattering medium in the form of a parallelepiped with a plane–parallel radiation flux incident normally on one of its sides. In Ref. 1 deposited in VINITI an algorithm for calculating the radiation fluxes in a parallelepiped with arbitrary ratio between edges was presented while in Ref. 4 – its particular case in which the edges transverse to an incident flux were identical.

The method of multiple reflections or, more exactly, the algorithm proposed by the author can be used to calculate the following seven characteristics: the ratio of the number of photons leaving the scattering medium through each of six sides to the total number of photons entering the scattering medium as well as the portion of the absorbed photons provided that the absorption takes place during the scattering event. In the particular case of parallelepiped of square cross section $L_y = L_z$ and medium without absorption the method yields the following three characteristics: the portion of photons outgoing through a back side I_1, the portion of photons reflected from a face I_2, and the portion of photons outgoing through sides I_3. These three quantities are subject to the obvious relation

$$I_1 + I_2 + I_3 = 1\ . \qquad (6)$$

When providing a theoretical foundation for this method, the authors assume in fact that in a scattering medium the quantities $I_1(x)$ and $I_2(x)$ having a meaning of radiation fluxes (5) along the x axis in forward and backward directions are described by the two–flux approximation, i.e., by the system of equations (3). Then for the preset coefficients α' and β' and the parallelepiped length L_x the elementary formulas representing a solution of the system of equations (3) are valid for the fluxes I_1 and I_2 outgoing from the front and back sides, respectively. There is no need to reproduce these formulas here.

It is evident that a photon loss through the parallelepiped sides is equivalent to some "effective" absorption of the photons moving along the x axis by the medium. Therefore, the coefficients α' and β' must be essentially dependent on the lateral dimensions of the parallelepiped L_y and on the shape of the scattering phase function $p(\mathbf{n}' \to \mathbf{n})$.

From conditional and vague considerations, when the medium was tentatively divided into a system of plates and bars and some relations of the particle number balance were duscussed in these systems, the author of Ref. 1 concluded that the coefficients α' and β' could be also found from the solution of equations of two–flux approximation (3) in which the parameters were the lateral dimensions of the parallelepiped L_y and new coefficients α'' and β'' determined as some integrals of the scattering phase function $p(\mathbf{n}' \to \mathbf{n})$. All the totality of formulas expressing the coefficients α', β', α'', and β'' in terms of the lateral dimensions of the parallelepiped L_y and the scattering phase function together with the standard formulas for the solution of equations of two–flux approximation (3) formed the algorithm of the method of multiple reflections.

Note that to obtain the solutions of equation (3) the authors of the method often used an iterative expansion which had a physical meaning of multiple reflections of photons from the layers of scattering medium. This fact explains the name of this method. However, such solutions can be easily obtained by standard methods with corresponding boundary conditions or radiation sources placed inside of the scattering medium, because from the mathematical point of view the system of equations (3) is a trivial system of differential equations with constant coefficients.

Hence in my opinion, the method of multiple reflections is nothing but a modification of the two–flux approximation developed by the authors for a scattering medium in the form of a parallelepiped when the radiation is incident normally on it.

2. ACCURACY OF THE METHOD

It seems that the authors of the method try to apply their calculations to as many practical problems as possible showing little concern for the validity of the method, estimate of its accuracy, and limits of its applicability. In those cases in which the accuracy was discussed in their papers they usually demonstrated a very good agreement with the results obtained by more regorous methods.

To compensate for this deficiency, I carried out the calculations by the algorithm described in Ref. 4 and compared the calculated results with the datae obtained by the Monte Carlo (MC) method by Davies in Ref. 8. Note that the authors of the method of multiple reflections (MR) often referred to this paper to compare the results of their calculations.

I used the formulas of the method of multiple reflections[4] for the simplest case of isotropic scattering in a medium without absorption. Instead of geometric parameters of the parallelepiped L_x and $L_y = L_z$, I used more suitable parameters, namely, the longitudinal (in the direction of incident radiation) optical thickness τ_x and the transverse optical thickness τ_y

$$\tau_x = \alpha L_x , \quad \tau_y = \alpha L_y .$$

TABLE I.

τ_x		τ_y 0.1 MC	MR	1 MC	MR	10 MC	MR	100 MC	MR	500 MC	MR
0.1	I_1	92	92	–	–	–	–	–	–	–	–
	I_2	2	2	–	–	–	–	–	–	–	–
	I_3	6	6	–	–	–	–	–	–	–	–
1	I_1	38	44	45	47	62	60	65	66	67	67
	I_2	2	9	13	11	30	26	34	32	33	33
	I_3	60	47	42	42	8	14	1	2	0	0
10	I_1	–	–	0	0	2	2	12	12	13	16
	I_2	–	–	13	14	50	43	81	74	85	81
	I_3	–	–	87	86	48	55	7	14	2	3
100	I_1	–	–	0	0	0	0	1	0	1	0
	I_2	–	–	13	14	50	43	88	77	95	89
	I_3	–	–	87	86	50	57	11	23	4	11
200	I_1	–	–	0	0	0	0	1	0	1	0
	I_2	–	–	13	14	50	43	88	77	95	89
	I_3	–	–	87	86	50	57	11	23	4	11

The values of I_1, I_2, and I_3 obtained from the formulas of the method of multiple reflections for indicated dimensions of parallelepiped are presented in Table I in the right columns (MR). Here the numbers I_1, I_2, and I_3 having the meanings of portions of the total number of photons are given in percent for clarity of representation. Before tabulating, the quantities I_1, I_2, and I_3, calculated from the formulas derived in Ref. 4, were rounded off with an error of not more than 1%, given that the condition of normalization (6) was satisfied.

The results of the Davies calculations by the Monte Carlo method of the same quantities I_1, I_2, and I_3 were presented in Ref. 8 in the form of nomograms. In the left columns of Table I (MC) I present the quantities retrieved from these nomograms also with an error of not more than 1%. The dashes in the squares of the table point to the fact that it was impossible to retrieve from the nomograms all the three quantities with satisfactory accuracy.

The Davies data can be assumed correct in comparison with the method of multiple reflections. For clarity of representation of the accuracy of the method of multiple reflections I calculated the systematic rms errors from the following formula:

$$\eta_i = \frac{|I'_i - I_i|}{I'_i} \cdot 100\% ,$$

where I'_i are the data obtained by the Monte Carlo method and I_i are the results obtained by the method of multiple reflections. The rms errors are tabulated in Table II.

TABLE II.

τ_x		τ_y 0.1	1	10	100	500
0.1	η_1	0	–	–	–	–
	η_2	0	–	–	–	–
	η_3	0	–	–	–	–
1	η_1	16	4	3	2	0
	η_2	350	15	13	6	0
	η_3	22	0	75	100	0
10	η_1	–	0	0	0	23
	η_2	–	8	14	9	5
	η_3	–	1	15	100	50
100	η_1	–	0	0	100	100
	η_2	–	8	14	12	6
	η_3	–	1	14	109	175
200	η_1	–	0	0	100	100
	η_2	–	1	14	12	6
	η_3	–	1	14	109	175

Comparing the data of Tables I and II, it can be concluded that the rms errors of the formulas of the method of multiple reflections are, on the average, about 10% when the quantities I_i are sufficiently large. For small values of fluxes I_i the method of multiple reflections gives the results which differ from the exact values by two or three times.

3. FEASIBILITY OF THE METHOD

Feasibility of the method of multiple reflections is of great importance. At the first glance, the main barrier to the application of this method in practice is the geometry of scattering medium —— parallelepiped. It is hardly possible to develop an analogous and at the same time sufficiently simple algorithm for scattering media of different geometry.

But there is another much more important obstacle to the practical application of this method lying in the fact that these three quantities I_1, I_2, and I_3 are of no practical significance.

To prove this statement let us turn to Fig. 1, in which the scattering medium is shown by the parallelepiped A. The question arises: what measured values correspond to the three quantities I_1, I_2, and I_3? Since the quantities I_i are the radiation fluxes, they can be measured by a detector being close to the scattering medium and covering some side of the parallelepiped. If the detector size is less than the parallelepiped dimensions, the same result can be obtained by scanning the side of the parallelepiped A.

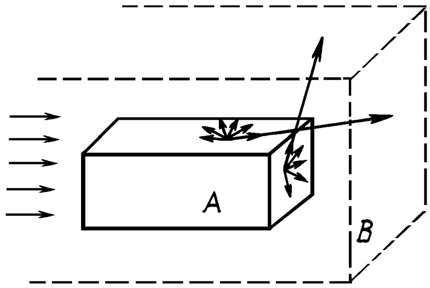

FIG. 1.

Now let us imagine the parallelepiped B enclosing the parallelepiped A and having much larger dimensions and carry out the same measurements for the sides of the parallelepiped B. Photons leaving the surface of the scattering medium (parallelepiped A) can leave any point of the surface A and move in any direction. It can be seen from the figure that the photon fluxes moving through the sides of the parallelepiped B will differ from the previous quantities I_i. It is clear that the method of multiple reflections cannot estimate these new quantities I_i'' which will depend on the distance to the scattering medium and dimensions of the parallelepiped B.

It is difficult to imagine a situation in which the measurements of fluxes close to the scattering medium rather than at some distance from it are necessary. For example, calculations of multiply scattered radiation in scattering media in the form of parallelepiped were carried out by the Monte Carlo method to estimate the transmission of the solar radiation by cumulus clouds.[8,9] Irradiation of each cloud by the neighboring clouds was also considered in Ref. 9. The quantities I_i obtained by the method of multiple reflections are inapplicable because here it is necessary to take into account at least the distance between the clouds.

As to an individual cloud, in this case the photon fluxes through the planes $x = \text{const}$ located before the scattering medium or after it are of practical significance. These constants which are referred to as the transmission coefficient κ_1 and reflection coefficient κ_2 satisfy the relation

$$\kappa_1 + \kappa_2 = 1 .$$

The fluxes I_i obtained by the method of multiple reflections are very uncertainly related to these coefficients. Let us consider, for example, a cube of the scattering medium with the optical thicknesses $\tau_x = \tau_y = 1$. According to Table I, 42% of photons leave this cube through its sides. The method of multiple reflections leaves arbitrariness in the distribution of 42% of photons between the transmission κ_1 and reflection κ_2 coefficients.

Thus, the feasibility of the method of multiple reflections is highly problematic.

REFERENCES

1. B.A. Savel'ev, "*Method of multiple reflections in problems of optical radiative transfer in media with uniformly distributed sources*", VINITI, No. 547–78, Moscow, 1978, 52 pp.
2. B.V. Goryachev, M.V. Kabanov, and B.A. Savel'ev, Atm. Opt. **3**, No. 2, 125–132 (1990).
3. B.V. Goryachev, M.V. Kabanov, and B.A. Savel'ev, Atm. Opt. **4**, No. 8, 581–586 (1991).
4. B.V. Goryachev, M.V. Kabanov, S.B. Mogil'nitskii, et al., Atm. Opt. **4**, No. 8, 587–588 (1991).
5. V.I. Klyatskin, *Stochastic Equations and Waves in Randomly Inhomogeneous Media* (Nauka, Moscow, 1980), 336 pp.
6. A. Isimaru, *Propagation and Scattering of Waves in Randomly Inhomogeneous Media,* Vol. 1 [Russian translation] (Mir, Moscow, 1981), 280 pp.
7. É.P. Zege, A.P. Ivanov, and I.L. Katsev, *Image Transfer through Scattering Medium* (Nauka i Tekhnika, Minsk, 1985), 327 pp.
8. R. Davies, J. Atmos. Sci. **35**, No. 9, 1712–1725 (1978).
9. M. Aida, J. Quant. Spectrosc. Radiat. Transfer **17**, No. 3, 303–310 (1977).