STATISTICAL MOMENTS OF THE IMAGING SYSTEM PARAMETERS IN THE TURBULENT ATMOSPHERE

S.E. Skipetrov and S.S. Chesnokov

M.V. Lomonosov State University, Moscow

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The first and second statistical moments of the optical transfer function and those of integral resolution of the imaging system in the turbulent atmosphere are calculated in the approximation of the Markovian process. The parameters of the optical imaging system under certain conditions may undergo strong fluctuations which can be caused by even weak fluctuations of wave parameters.

INTRODUCTION

The imaging systems under incoherent illumination are usually described by the optical transfer function (OTF) $\tau(\Omega)$ which connects the angular spectrum $I(\Omega)$ of an image formed with the presence of random-inhomogeneous medium with the angular spectrum of the ideal geometric optical image $I_g(\Omega)$:$I(\Omega) = \tau(\Omega) \times I_g(\Omega)$ (Ref. 1). Thus, OTF plays the part of a spatial frequency filter which describes how the angular spectrum of the image varies under the effect of diffraction at the aperture of the optical system and inhomogeneities of dielectric permeability in the medium. The passband of such a filter is characterized by the integral of $\tau(\Omega)$ over all the angular frequencies. The integral is said to be the integral resolution of the optical system.

The influence of atmospheric turbulence on mean values of OTF and integral resolution is well studied for both turbulence with the Kolmogorov fluctuation spectrum of the refractive index[2] and turbulence described by spectra with finite internal[3,4] and external[5,6] scales. However, characteristics of fluctuations of these parameters near mean values are also of interest, especially, the variance of OTF and integral resolution fluctuations, and correlation of OTF values at different angular frequencies.

In this paper, the mean value $<\tau(\Omega)>$ and correlation function $K(\Omega_1, \Omega_2) = <\tau(\Omega_1)\tau^*(\Omega_2)>$ of the OTF are calculated theoretically in an approximation of the Markovian process. On the base of the expressions obtained, variance of fluctuations of OTF and integral resolution is calculated. In our opinion, the obtained results permit significant improvement of the understanding of the fluctuation structure for parameters describing the imaging system operation under the conditions of atmospheric turbulence.

1. CALCULATION OF OTF STATISTICAL MOMENTS

Let us consider a plane light wave propagated along the z axis in a turbulent medium occupying the half-space $z > 0$. In the scalar approximation, we describe the electric field strength in the wave by a function $u(\rho, z)$, where ρ is the vector in the plane perpendicular to the z axis. Then, as it is shown, for instance, in Refs. 7 and 8, the equations for the second and fourth order correlation functions Γ_2, Γ_4 of the scalar light field $u(\rho, z)$ can be obtained in the approximation of the Markovian process

$$\frac{\partial \Gamma_2}{\partial z} - \frac{i}{2k}[\Delta_1 - \Delta_2]\,\Gamma_2 + \frac{\pi k^2}{4} H(\rho_1 - \rho_2)\,\Gamma_2 = 0\ ; \tag{1}$$

$$\frac{\partial \Gamma_4}{\partial z} - \frac{i}{2k}[\Delta_1 + \Delta_2 - \Delta_3 - \Delta_4]\,\Gamma_4 + \frac{\pi k^2}{4} F(\rho_1, \rho_2, \rho_3, \rho_4)\,\Gamma_4 = 0\ , \tag{2}$$

where

$$\Gamma_2(\rho_1, \rho_2) = <u(\rho_1)\, u^*(\rho_2)>\ ; \tag{3}$$

$$\Gamma_4(\rho_1, \rho_2, \rho_3, \rho_4) = <u(\rho_1)\, u(\rho_2)\, u^*(\rho_3)\, u^*(\rho_4)>\ ; \tag{4}$$

$$H(\rho) = 2\int d^2\varkappa[1 - \cos(\varkappa\rho)]\,\Phi(\varkappa, 0)\ ; \tag{5}$$

$$F(\rho_1, \rho_2, \rho_3, \rho_4) = H(\rho_1 - \rho_3) + H(\rho_1 - \rho_4) + H(\rho_2 - \rho_3) + H(\rho_2 - \rho_4) - H(\rho_1 - \rho_2) - H(\rho_3 - \rho_4); \tag{6}$$

Δ_n is the Laplace operator with respect to the variable ρ_n; $\varkappa$ is the vector of spatial frequencies in the plane perpendicular to the direction of wave propagation z; $k = 2\pi/\lambda$ is the wave number of the considered plane wave; $\Phi_\varepsilon(\varkappa, \varkappa_z)$ is the power spectrum of dielectric permeability fluctuations. It is easy to see that the function $H(\rho)$ for isotropic and statistically homogeneous turbulence is related to the wave structure function $D(\rho)$ of a plane wave having passed a path z in the medium. The relation is simple: $H(\rho) = H(\rho) = 2/(\pi k^2 z)D(\rho)$.

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As follows from the definition of OTF,[1] the averaged OTF $<\tau(\Omega)>$ and correlation function $K(\Omega_1, \Omega_2) = <\tau(\Omega_1)\tau^*(\Omega_2)>$ can be expressed via Γ_2 and Γ_4, respectively, as the integrals

$$<\tau(\Omega)> = \frac{1}{A}\int d^2\rho P(\rho + \lambda\Omega/2)\, P^*(\rho - \lambda\Omega/2) \times$$

$$\times\, \Gamma_2(\rho + \lambda\Omega/2,\ \rho - \lambda\Omega/2)\ ; \quad (7)$$

$$K(\Omega_1, \Omega_2) = \frac{1}{A^2}\int d^2\rho' \int d^2\rho'' \times$$

$$\times P(\rho_1)\, P(\rho_2)\, P^*(\rho_3)\, P^*(\rho_4)\, \Gamma_4(\rho_1, \rho_2, \rho_3, \rho_4)\,, \quad (8)$$

where $\rho_1 = \rho' + \lambda\Omega_1/2$; $\rho_2 = \rho'' - \lambda\Omega_2/2$; $\rho_3 = \rho' - \lambda\Omega_1/2$; $\rho_4 = \rho'' + \lambda\Omega_2/2$; A is the scale constant; $P(\rho)$ is the pupil function of the optical system. Thus, multiplying Eqs. (1) and (2) by the corresponding pupil functions and integrating Eq. (1) over the variable ρ and Eq. (2) over the variables ρ' and ρ'', we obtain

$$\frac{\partial}{\partial z} <\tau(\Omega)> = \frac{\pi k^2}{4} H(\lambda\Omega)\, <\tau(\Omega)>\ ; \quad (9)$$

$$\frac{\partial}{\partial z} K(\Omega_1, \Omega_2) = -\frac{\pi k^2}{4}\, [H(\lambda\Omega_1) + H(-\lambda\Omega_2)] \times$$

$$\times\, K(\Omega_1, \Omega_2) - \frac{\pi k^2}{4}\int d^2\rho' \int d^2\rho'' \times$$

$$\times\, P(\rho_1)\ P(\rho_2)\ P^*(\rho_3)\ P^*(\rho_4) \times \big[H(\rho' - \rho'' +$$

$$+\, \lambda(\Omega_1 - \Omega_2)/2) + H(\rho'' - \rho' + \lambda(\Omega_1 - \Omega_2)/2) -$$

$$-\, H(\rho' - \rho'' + \lambda(\Omega_1 + \Omega_2)/2) -$$

$$-\, H(\rho' - \rho'' - \lambda(\Omega_1 + \Omega_2)/2)\big]\, \Gamma_4(\rho_1, \rho_2, \rho_3, \rho_4)\,. \quad (10)$$

As seen from Eq. (9), the averaged OTF $<\tau(\Omega)>$ depends on the absolute value Ω of the angular frequency Ω that stems from statistical homogeneity of the field $u(\rho, z)$ in the plane perpendicular to the direction of wave propagation. By the same reason, taking into account that the pupil function $P(\rho)$ decreases to zero as $|\rho|$ is sufficiently large, the differential terms $[\Delta_1 - \Delta_2]\, \Gamma_2$ and $[\Delta_1 + \Delta_2 - \Delta_3 - \Delta_4]\, \Gamma_4$ give no contribution when Eqs. (1) and (2) are integrated.

The expression (10) is not a closed equation with respect to $K(\Omega_1, \Omega_2)$, because the coherence function of the fourth order Γ_4 enters into it explicitly. At the same time, Eq. (9) can be easily solved regardless a concrete form of the spectrum $\Phi_\varepsilon(\varkappa)$ and, taking into account the evident initial condition $<\tau(\Omega)> = \tau_0(\Omega)$ for $z = 0$, it leads to the expression for the average OTF . This expression coincides with that obtained earlier in another way[2]:

$$<\tau(\Omega)> = \tau_0(\Omega)\, \exp\left\{-\frac{\pi k^2 z}{4} H(\lambda\Omega)\right\} \equiv$$

$$\equiv \tau_0(\Omega)\, \exp\left\{-\frac{1}{2} D(\lambda\Omega)\right\}, \quad (11)$$

where $\tau_0(\Omega)$ is the OTF of a diffraction-bounded imaging system.[1]

1.1. The case of a small receiving aperture *($d_0 << l_0$)*

If the aperture diameter d_0 of the considered optical system is much less than the inner turbulence scale l_0, then, within the aperture limits, we can assume that $D(\rho) = 3.28\, C_n^2\, k^2\, z\, l_0^{-1/3}\, \rho^2$ (Ref. 7) and $H(\rho) = 2.088\, C_n^2\, l_0^{-1/3}\, \rho^2$, what leads to the following expression for the average OTF:

$$<\tau(\Omega)> = \tau_0(\Omega)\, \exp\left\{-\, (\Omega^2/\Omega_c^2\right\}, \quad (12)$$

where $\Omega_c^2 = 0.015\, l_0^{1/3}\, C_n^{-2}\, z^{-1}$. This result coincides with that discussed in Ref. 3.

Since H is a quadratic function of ρ, the expression (10) can be transformed into a closed equation with respect to K for $d_0 \ll l_0$; indeed, the multiplier in the square brackets under the integral sign in Eq. (10) becomes independent of the integration variables ρ' and ρ'' and can be factored outside the integral sign:

$$\frac{\partial}{\partial z} K(\Omega_1, \Omega_2) =$$

$$= 2.088\, \pi^3\, C_n^2\, l_0^{-1/3}\, (\Omega_1 - \Omega_2)^2\, K(\Omega_1, \Omega_2)\ . \quad (13)$$

This equation can be easily solved and, taking into account the evident initial condition $K(\Omega_1, \Omega_2) = \tau_0(\Omega_1)\, \tau_0(\Omega_2)$, leads to the following expression:

$$K(\Omega_1, \Omega_2) = \tau_0(\Omega_1)\tau_0(\Omega_2)\, \exp\left\{-\frac{(\Omega_1 - \Omega_2)^2}{\Omega_c^2}\right\}. \quad (14)$$

As follows from Eqs. (12), (14) for $\Omega_1 = \Omega_2 = \Omega$, $<|\tau(\Omega)|^2> = \tau_0^2(\Omega)$, $<\tau(\Omega)>^2 = \tau_0^2(\Omega)\exp\{-2\Omega^2/\Omega_c^2\}$. So the variance of OTF fluctuations σ_τ^2 depends on only the absolute value Ω of angular frequency and equals

$$\sigma_\tau^2 \equiv <|\tau(\Omega)|^2> - <\tau(\Omega)>^2 =$$

$$= \tau_0^2(\Omega)\, [1 - \exp\{-2(\Omega^2/\Omega_c^2)\}]\ . \quad (15)$$

To make the further analysis more instructive, let us assume that the OTF $\tau_0(\Omega)$ is Gaussian:

$$\tau_0^2(\Omega) = \exp\{-(\Omega^2/\Omega_0^2)\}\,, \quad (16)$$

where $\Omega_0 = d_0/\lambda$; d_0 is the «effective[diameter of the optical system aperture. Substitution of such an expression for $\tau_0(\Omega)$ into Eq. (15) leads to the following behavior of variance of OTF fluctuations: σ_τ^2 equals zero for $\Omega = 0$, then it monotonically increases to its maximal value

$$(\sigma_\tau^2)_{\max} = (\Omega_0^2/\Omega_c^2)[1 + (\Omega_0^2/\Omega_c^2)]^{-(1 + \Omega_c^2/\Omega_0^2)} =$$

$$= \begin{cases} e^{-1}\, \Omega_0^2/\Omega_c^2, & \Omega_0 \ll \Omega_c, \\ 1, & \Omega_0 \gg \Omega_c, \end{cases} \quad (17)$$

reached at $\Omega_1^2 = (\Omega_c^2/2) \ln[1 + \Omega_0^2/\Omega_c^2]$; then it monotonically decreases with further increase of Ω. As seen from Eq. (17), for large relative aperture dimensions ($\Omega_0 > \Omega_c$), the value of fluctuation variance $\tau(\mathbf{\Omega})$ can be of the order of unit at the angular frequency Ω_1. It means that OTF $\tau(\mathbf{\Omega})$ undergoes strong fluctuations even in the case when fluctuations of wave phase and logarithm of wave amplitude may be small.

1.2. Approximate analysis for the case $l_0 \ll d_0 \ll L_0$

In the case when the aperture diameter d_0 exceeds the internal turbulence scale l_0 and remains small as compared with the outer scale L_0, we have, according to Ref. 7, $D(\rho) = 6.88(\rho/r_0)^{5/3}$ and $H(\rho) = 2/(\pi k^2 z)\; 6.88\; (\rho/r_0)^{5/3}$, where $r_0 = 0.185[\lambda^2/(C_n^2\, z)]^{3/5}$ is the Fried radius.[2] The direct substitution of the expression for $H(\rho)$ into Eq. (10) does not permit us to obtain a closed equation with respect to $K(\mathbf{\Omega}_1, \mathbf{\Omega}_2)$. However, if the aperture diameter d_0 is still not very long, the function $H(\rho)$ in the integrand of Eq. (10) can be approximated, with sufficient accuracy, by the quadratic function $B\rho^2$, where the coefficient $B \simeq 13.76/(\pi\, k^2\, z\, r_0^{5/3}\, d_0^{1/3})$ is obtained by the least squares method under the assumption that the aperture of the optical system is a circle of diameter d_0. This makes it possible to obtain an approximate equation for $K(\mathbf{\Omega}_1, \mathbf{\Omega}_2)$ similar to Eq. (13). Its solution yields

$$K(\mathbf{\Omega}_1, \mathbf{\Omega}_2) \simeq T_0(\Omega_1)\, T_0(\Omega_2) \times$$

$$\times \exp\left\{-3.44\, (d_0/r_0)^{5/3} \left[(\Omega_1/\Omega_0)^{5/3} + \right.\right.$$

$$\left.\left. + (\Omega_2/\Omega_0)^{5/3} - 2\, (\mathbf{\Omega}_1/\Omega_0)\, (\mathbf{\Omega}_2/\Omega_0)\right]\right\}, \quad (18)$$

where $\Omega_0 = d_0/\lambda$ and

$$T_0(\Omega) = \frac{2}{\pi}\left[\arccos(\Omega/\Omega_0) - (\Omega/\Omega_0)\sqrt{1 - (\Omega/\Omega_0)^2}\right]. \quad (19)$$

The equation for variance of OTF fluctuations $\tau(\Omega)$ takes the form

$$\sigma_\tau^2 \simeq T_0^2(\Omega) \exp\left\{-6.88\, (d_0/r_0)^{5/3}\, (\Omega/\Omega_0)^{5/3}\right\} \times$$

$$\times \left[\exp\{2(\Omega^2/\Omega_0^2)\} - 1\right]. \quad (20)$$

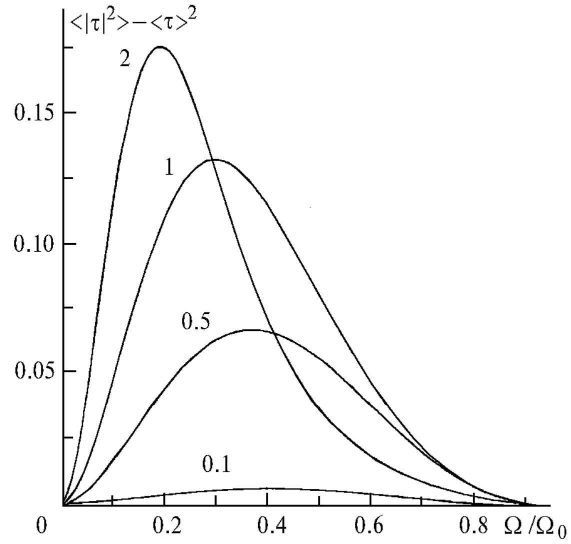

FIG. 1. Variance of OTF fluctuations σ_τ^2 as a function of the absolute value Ω of the angular frequency $\mathbf{\Omega}$ for the case $l_0 \ll d_0 \ll L_0$ at $d_0/r_0 = 0.1$, 0.5, 1, and 2.

In this case (see Fig. 1), the behavior of σ_τ^2 is qualitatively similar to that described by Eq. (15). However, the use of Eq. (20) is justified only for not very large aperture diameters d_0. As seen from Fig. 1, variances of OTF fluctuations σ_τ^2 is rather large, although it does not reach unity for the chosen values of the ratio d_0/r_0.

2. CALCULATION OF INTEGRAL RESOLUTION MOMENTS

Let us now consider the statistics of integral resolution of the optical system under atmospheric turbulence, namely, its mean value

$$\langle R \rangle = \int d^2\, \mathbf{\Omega}\, \langle \tau(\Omega) \rangle\,, \quad (21)$$

mean value of its module squared

$$\langle |R|^2 \rangle = \int d^2\, \mathbf{\Omega}_1 \int d^2\, \mathbf{\Omega}_2\, K(\mathbf{\Omega}_1, \mathbf{\Omega}_2) \quad (22)$$

and fluctuation variance $\sigma_R^2 = \langle |R|^2 \rangle - \langle R \rangle^2$.

2.1. The case of a small receiving aperture ($d_0 \ll l_0$)

In this case, as above, we consider that $\tau_0(\Omega) = \exp\{-\Omega^2/\Omega_0^2\}$. Substituting Eqs. (11) and (14) into Eqs. (21) and (22), and using the equations from the Ref. 9, we obtain

$$\langle R \rangle = \frac{\pi\Omega_0^2}{1 + \Omega_0^2/\Omega_c^2} = \begin{cases} \pi\Omega_0^2\,, & \Omega_0 \ll \Omega_c, \\ \pi\Omega_c^2\,, & \Omega_0 \gg \Omega_c, \end{cases} \quad (23)$$

$$<|R|^2> = \frac{\pi^2\Omega_0^4}{1 + 2\,\Omega_0^2/\Omega_c^2} = \begin{cases}\pi^2\,\Omega_0^4, & \Omega_0 \ll \Omega_c,\\ \pi^2\,\Omega_0^2\,\Omega_c^2/2, & \Omega_0 \gg \Omega_c,\end{cases} \quad (24)$$

$$\sigma_R^2 = <R>^2 \frac{\Omega_0^4/\Omega_c^4}{1 + 2\,\Omega_0^2/\Omega_c^2} = \begin{cases}\pi^2\,\Omega_0^8/\Omega_c^4, & \Omega_0 \ll \Omega_c,\\ \pi^2\,\Omega_0^4\,\Omega_c^2/2, & \Omega_0 \gg \Omega_c.\end{cases} \quad (25)$$

As seen from Eq. (23), the mean integral resolution tends to a finite limit with the increase of the aperture diameter. The limit is determined by turbulence parameters (the value of Ω_c) what coincides with the conclusions of Ref. 4. At the same time, variance of fluctuations of integral resolution σ_R^2 unboundedly increases with the increase of the aperture diameter. This result, however, is not of practical importance because Eq. (25) is valid only for $d_0 \ll l_0$, and the increase of d_0 to values larger than the inner turbulence scale makes Eq. (25) inapplicable to the real physical situation.

2.2. Approximate analysis for the case $l_0 \ll d_0 \ll L_0$

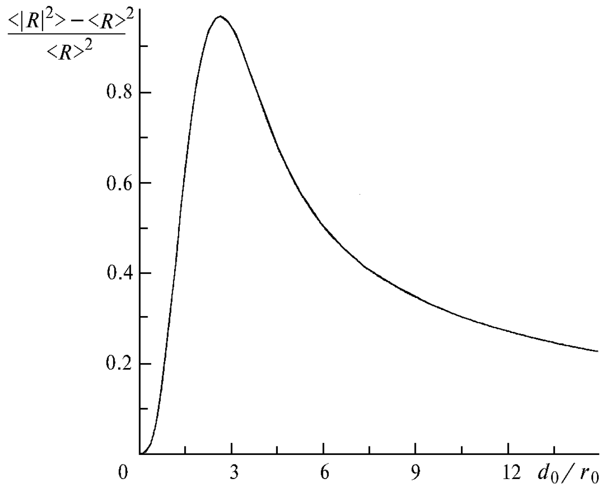

FIG. 2. Relative variance of fluctuations of integral resolution $\sigma_R^2/<R>^2$ as a function of the aperture diameter d_0 normalized to the Fried radius r_0.

If the aperture diameter is larger than the inner scale of turbulence but much less than the outer scale, Eqs. (11), (18), and (21), (22) permit only numerical calculation of the values $<R>$, $<|R|^2>$, and σ_R^2. Since the mean value of integral resolution was already analyzed,[1,2] we present the results of numerical integration for variance of fluctuations of integral resolution. Figure 2 presents the value $\sigma_R^2/<R>^2$ as a function of the aperture diameter d_0 normalized by Fried's radius r_0. The maximal value $\sigma_R^2/<R>^2 \sim 1$ is reached for $d_0/r_0 \simeq 2.6$. Thus, the integral resolution R for $d_0/r_0 \sim 3$ undergoes strong fluctuations which can be caused by weak fluctuations of the wave parameters (phase and log-amplitude). It means that instantaneous images can be both much worse and considerably better in their quality as compared with long-exposure ones. So, choosing the aperture diameter so that it exceeds the Fried radius 2–3 times, one can expect that, among some instantaneous object images, at least one is not distorted by turbulence.

CONCLUSION

Thus, using the approximation of the Markovian process, we calculated the mean values and second moments of OTF and integral resolution of an optical system in the turbulent atmosphere. It is established that OTF fluctuations are not similar at different angular frequencies; fluctuation maximum is reached at a certain frequency Ω_1 depending on the turbulence parameters and the optical system characteristics. Fluctuations of OTF at angular frequencies close to Ω_1 are rather high and can reach unity even under small fluctuations of the light wave parameters.

Calculations of the moments of integral resolution R of the optical system demonstrate that variance of resolution fluctuations significantly depends on the aperture diameter d_0. For $d_0 \ll l_0$, variance monotonically grows with the increase of d_0, then, as shown by the approximate analysis of the case $l_0 \ll d_0 \ll L_0$, it reaches the absolute maximum at $d_0 \sim 3r_0$ and monotonically decreases to zero with further increase of the aperture diameter. Our calculations permit the estimation of the maximal fluctuation variance of R as close to unity. This indicates that strong fluctuations of integral resolution of the optical system are observed for the aperture diameter close to $3r_0$. They may be caused by weak fluctuations of the optical wave parameters.

It should be also noted that, if the distribution law for the random parameter R is assumed to be Gaussian, our calculations of its mean value $<R>$ and variance of fluctuations σ_R^2 permit us to estimate the probability to obtain an instantaneous OTF realization characterized by the integral width R larger than a certain given value. In particular, this yields the probability to obtain $R > <R>$ equal to 1/2.

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