To an analysis of the acoustic Doppler effect in a three-dimensional inhomogeneous moving medium

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Received December 10, 1998

For the first time the Fermat variational principle, known in geometric optics, is used to analyze the acoustic Doppler effect. On its basis, exact and physically vivid analytical formulas have been derived for the Doppler effect in the general case of sound propagating in a three-dimensional inhomogeneous moving medium.

Introduction

The Doppler effect, consisting in the difference between the recorded frequency of oscillations ω_r and the frequency of oscillations emitted by a source ω_s when the source or the receiver moves, is observed for all wave phenomena – optical, acoustic, and others. Its treatment depends on whether one can take into account only the velocity of *relative motion* of the source and the receiver $\mathbf{v}$ or the velocities of motion of the source $\mathbf{w}$ and the receiver $\mathbf{u}$ *relative to the medium* should be considered.[1–3]

For sound waves, undoubtedly, the second case is true: they can propagate only in a material medium (for example, in a gas), and the velocities of motion of the source and receiver are always considered separately. In particular, the classical formula for the acoustic effect in a homogeneous stationary medium has the form[1,2]

$$\omega_r = \omega_s \frac{1 - \mathbf{n}\cdot\mathbf{u}/c}{1 - \mathbf{n}\cdot\mathbf{w}/c}, \tag{1}$$

where c is the velocity of sound propagation in the stationary medium, and $\mathbf{n}$ is the normal to the wavefront. From Eq. (1) it follows that in acoustics the formulas for the Doppler effect differ for the stationary source ($\mathbf{w} = 0$) and the stationary receiver ($\mathbf{u} = 0$). Therefore, measurements of the acoustic Doppler frequency shift, in principle, allow one to judge not only the velocity of relative motion of the source and receiver $\mathbf{v} = \mathbf{w} - \mathbf{u}$, but also the velocities of motion of the source and receiver relative to the medium ($\mathbf{w}$ and $\mathbf{u}$). Moreover, if in Eq. (1) we proceed to new coordinates, in which the medium moves with velocity $\mathbf{v}$, the formula for the acoustic Doppler effect also will include $\mathbf{v}$, that is, the movement of the medium can be detected directly from measurements of the frequency ω_r.

Before the advent of the special relativity theory (SRT) it was believed that the essence of the Doppler effect for the electromagnetic waves did not differ from the analogous phenomenon for sound. Moreover, it was considered that the basic electrodynamic equations were valid only in one inertial system of coordinates named the absolute reference system, stationary relative to global ether, which was understood as a special medium – the carrier of the electromagnetic processes, which fills in the whole space and any matter. Negative result of the well-known experiments on detection of an ethereal wind and other experimental data confirming consequences of the SRT abandoned the hypothesis of global ether. From the relativistic viewpoint, the Doppler formula for the electromagnetic waves should include only the velocity of relative motion of the source and receiver $\mathbf{v}$, that is, it should be independent of the choice of the inertial system of coordinates. For example, in optics the formula for the Doppler effect in the vacuum is usually written as[1,2]

$$\omega_r = \omega_s \frac{\sqrt{1 - \upsilon^2/c_e^2}}{1 - \boldsymbol{\upsilon}\cdot\mathbf{s}/c_e}, \tag{2}$$

where c_e is the velocity of light in the vacuum, and $\mathbf{s}$ is the unit vector tangential to an optical ray.

The well-known facts have been stated above to emphasize the principal importance of consideration of possible movement of the medium in the formulas for the acoustic Doppler effect. The situation is complicated by the fact that the movement of realistic media, in which the sound waves can propagate, is nonuniform very frequently. For example, in the atmosphere the wind velocity depends on the coordinates and time. Thus, it appears impossible to find the absolute coordinate system, stationary relative to the entire region of the medium influencing the sound propagation. Even a case is possible, in which the sound source and the receiver are completely entrained by the moving medium and, hence, each of them is stationary relative to it; nevertheless, owing to their motion relative to each other, the Doppler effect may take place.

The Doppler effect for the sound waves propagating in the inhomogeneous moving medium was examined in Refs. 8–14. Unlike these works, below the variational principle is used for the first time for this purpose, well-known in mechanics and referred to as the Fermat principle in geometric optics. As demonstrated, this approach allows exact and physically vivid formulas to be obtained for the Doppler effect in the general case of sound propagation in the inhomogeneous moving medium.

General solution

Let the point-size sound source and receiver move with velocities **w** and **u** in the continuous inhomogeneous medium also moving with nonuniform subsonic velocity $\mathbf{v}(\mathbf{r})$. The positions of the source and receiver in the coordinate system L, in which the velocities **w**, **u**, and $\mathbf{v}(\mathbf{r})$ are specified, will be described by variable radius-vectors $\mathbf{r}_s(t)$ and $\mathbf{r}_r(t)$. Let us also assume that the source emits harmonic oscillations, whose phase in the moving coordinate system L', affixed to it, is described by the function of time $\Phi_s(t') = \omega_s t' + \Phi_0$, where $\omega_s = \partial\Phi_s(t')/\partial t'$ is the circular frequency of these oscillations (in the coordinate system L').

It is obvious that the wave perturbation with the given phase, originated at time t_s at the point of sound emission $\mathbf{r}'_r = \mathbf{r}'_s(t'_r)$, for the reason of finite velocity of its propagation, will reach the point of reception $\mathbf{r}'_r = \mathbf{r}'_r(t')$ in time $\tau' = t' - t'_s \neq 0$. Therefore, $\Phi'(\mathbf{r}'_r, t') = \Phi_s(t' - \tau')$, and hence using the expression for $\Phi_s(t')$, we can write

$$\Phi'(\mathbf{r}'_r, t') = \Phi_s(t') - \omega_s \tau'. \tag{3}$$

Let us take advantage of the phase invariance of the same wave in Eq. (3) for systems of coordinates L and L' (See Refs. 1 and 2), expressed by the equality

$$\Phi(\mathbf{r}_r, t) = \Phi'(\mathbf{r}'_r, t'), \tag{4}$$

where $\mathbf{r}_r$, t and $\mathbf{r}'_r$, t' are the coordinates and times of the same event in coordinate systems L and L', respectively, related by the Galilean transformation in acoustics.

For the Galilean transformation, $t = t'$ and hence $\tau = \tau'$, where $\tau = t - t_s$ is the time of sound propagation from the point $\mathbf{r}_s(t_s)$ to the point $\mathbf{r}_r(t)$ in the stationary system of coordinates L. Therefore, substituting Eq. (3) into Eq. (4), we obtain

$$\Phi(\mathbf{r}_r, t) = \Phi_s(t) - \omega_s \tau. \tag{5}$$

The frequency of sound oscillations, recorded by the receiver (in the coordinate system L'', moving with the receiver with the velocity **u**), can be found from the formula

$$\omega_r = \partial\Phi''[\mathbf{r}_r(t'')]/\partial t'', \tag{6}$$

whose applicability limits were examined in Ref. 7. Here, they were taken into account in problem formulation.

By virtue of invariance of the wave phase, on account of the formulas for the Galilean transformation, in Eq. (6) $\Phi''[\mathbf{r}_r(t'')] = \Phi(\mathbf{r}_r, t)$ and $t'' = t$, that is, $\partial\Phi''[\mathbf{r}_r(t'')]/\partial t'' = \partial\Phi(\mathbf{r}_r, t)/\partial t$. As a result, on account of Eq. (5), the general formula for the Doppler effect in acoustics follows from Eq. (6):

$$\omega_r = \omega_s \{1 - \partial\tau[\mathbf{r}_s(t_s), \mathbf{r}_r(t)]/\partial t\}, \tag{7}$$

which imposes no restrictions on the character of inhomogeneities in the medium and considers the functional dependence of τ on the coordinates of the source and the receiver.

Formula (7) demonstrates that the Doppler frequency shift of the received sound oscillations is caused only by changes in the time of sound signal propagation (or in the wave energy) from the source to the receiver.

Derivation of the formulas for the acoustic Doppler effect

Let the medium satisfies to the condition of applicability of the geometric acoustics equations $\lambda \ll a$, where λ is the acoustic wavelength, and a is the characteristic size of inhomogeneities of the medium. In this case, the time of sound propagation τ from one point $M_1(x_1, y_1, z_1)$ of the Cartesian space (x, y, z) to the other point $M_2(x_2, y_2, z_2)$ can be obtained from the formula

$$\tau = \int_{M_1}^{M_2} \frac{dl}{U(M)} \tag{8}$$

known in geometric acoustics (see, for example, Ref. 13). Here, the integration is carried out along the sound ray connecting points M_1 and M_2, dl is the element of arc length along the ray, $U = |c\mathbf{n} + \mathbf{v}|$ is the modulus of the group velocity of sound in the coordinate system L (Ref. 7), c is the velocity of sound in the stationary medium, and **n** is the unit vector orthogonal to the wavefront. Formula (8) is valid for the general case of three-dimensional inhomogeneous moving medium. When the sound ray behavior is described by the parametric equations

$$x = x(\sigma), \quad y = y(\sigma), \quad z = z(\sigma),$$

Eq. (8) also can be written as

$$\tau = \int_{M_1}^{M_2} \frac{\sqrt{(x')^2 + (y')^2 + (z')^2}}{U(x, y, z)}\, d\sigma, \tag{9}$$

where $x' = \partial x/\partial\sigma$; $y' = \partial y/\partial\sigma$; $z' = \partial z/\partial\sigma$.

Curvilinear integrals (8), in which the integrand F can be represented as in Eq. (9), that is, in the form $F = F(x, y, z, x', y', z')$, in variational calculus are referred to as functionals dependent on the integration contour. For each specific trajectory from M_1 to M_2 they give specific numerical value of τ (Ref. 15). It is well known that the sound rays in the refracting medium satisfy the Fermat principle.[16] According to this principle, the ray connecting the points M_1 and M_2, should coincide with the curve M_1M_2, for which functional (9) gives the least value of τ. In other words, the sound ray is the extremal of functional (9), described by the Euler equations[15,17]

$$\frac{\partial}{\partial\sigma}\left(\frac{\partial F}{\partial x'}\right) - \frac{\partial F}{\partial x} = 0,\ \frac{\partial}{\partial\sigma}\left(\frac{\partial F}{\partial y'}\right) - \frac{\partial F}{\partial y} = 0,$$

$$\frac{\partial}{\partial\sigma}\left(\frac{\partial F}{\partial z'}\right) - \frac{\partial F}{\partial z} = 0,$$

where F is the integrand of Eq. (9).

If we recall Eq. (7), we easily notice that formula (9) can be used to analyze the Doppler effect in the inhomogeneous moving medium, when it is considered as a functional whose end points $M_1 = M_s[x_s(t_s), y_s(t_s), z_s(t_s)]$ and $M_2 = M_r[x_r(t), y_r(t), z_r(t)]$ with the coordinates coinciding with those of the source and receiver at the moments of sound emission t_s and reception t, respectively, are functions of time. For this problem it is convenient to introduce the curvilinear orthogonal system of coordinates[17] (x^0, y^0, z^0), in which one coordinate axis, for example x^0, coincides at each point with the ray connecting M_s and M_r, and two others (y^0 and z^0) intersect it and each other perpendicularly at any point of the ray. This coordinate system is feasible, because all the extremals coming from a fixed point M_0, form in the three-dimensional space the family of curves which do not intersect, except at the point M_0, that is, the family of extremals has only one extremal that passes through the given point[15] $M \neq M_0$. As a consequence, any point $M(x, y, z)$ in the vicinity of ray M_sM_r is an unambiguous function of x_0, y_0, z_0.

In the new system of coordinates, Eq. (9) can be written as

$$\tau = \int\limits_{x_s^0(t_s)}^{x_r^0(t)} \frac{\sqrt{1 + (\partial y^0/\partial x^0)^2 + (\partial z^0/\partial x^0)^2}}{U(x^0, y^0, z^0)}\, dx^0, \quad (10)$$

where $x_s^0(t)$ and $x_r^0(t)$ are the x^0 coordinates of the points M_s and M_r, respectively. Applying to Eq. (10) the general formula for the first variation of the functional whose ends are movable,[15,17] we obtain

$$\delta\tau = \left[\frac{\partial\tau}{\partial x^0}\,\delta x^0 + \frac{\partial\tau}{\partial y^0}\,\delta y^0 + \frac{\partial\tau}{\partial z^0}\,\delta z^0\right]\Bigg|_{x^0 = x_s^0(t_s)}^{x^0 = x_r^0(t)} +$$

$$+ \int\limits_{x_s^0(t_s)}^{x_r^0(t)} \left\{\left[\frac{\partial F}{\partial y^0} - \frac{\partial}{\partial x^0}\left(\frac{\partial F}{\partial y^0}\right)\right]\delta y^0 + \left[\frac{\partial F}{\partial z^0} - \frac{\partial}{\partial x^0}\left(\frac{\partial F}{\partial z^0}\right)\right]\delta z^0\right\} dx^0, \quad (11)$$

where F is the integrand of Eq. (10), and $Q\Big|_{x_1}^{x_2}$ denotes, as usually, the double substitution, that is, $Q\Big|_{x_1}^{x_2} = Q(x_2) - Q(x_1)$.

The integral term in Eq. (11) is identically equal to zero, because the integral is taken over the extremal. Therefore, on account of the formula[13] $\tau = \psi[\mathbf{r}_s(t_s), \mathbf{r}_r(t)]/c_0$, which expresses τ in terms of the increment of the eikonal ψ along the ray M_sM_r (here, c_0 is the characteristic value of the sound velocity c in the medium, for example, $c_0 = c(0)$), from Eq. (11) we obtain

$$\delta\tau = \frac{1}{c_0}\left[\frac{\partial\psi}{\partial x^0}\,\delta x^0 + \frac{\partial\psi}{\partial y^0}\,\delta y^0 + \frac{\partial\psi}{\partial z^0}\,\delta z^0\right]\Bigg|_{x^0 = x_s^0(t_s)}^{x^0 = x_r^0(t)}. \quad (12)$$

Taking into account that the first variation of the functional is the main (linear) part of its increment, and assuming that the coordinates of the mobile points $M_r(x_r^0, y_r^0, z_r^0)$ and $M_s(x_s^0, y_s^0, z_s^0)$ at the end depend solely on the parameter t, from Eq. (12) we derive

$$\frac{\partial\tau}{\partial t} = \frac{1}{c_0}\left\{\frac{\partial\psi}{\partial x_r^0}\frac{\partial x_r^0}{\partial t} + \frac{\partial\psi}{\partial y_r^0}\frac{\partial y_r^0}{\partial t} + \frac{\partial\psi}{\partial z_r^0}\frac{\partial z_r^0}{\partial t} - \right.$$

$$\left. - \frac{\partial\psi}{\partial x_s^0}\frac{\partial x_s^0}{\partial t_s}\frac{\partial t_s}{\partial t} - \frac{\partial\psi}{\partial y_s^0}\frac{\partial y_s^0}{\partial t_s}\frac{\partial t_s}{\partial t} - \frac{\partial\psi}{\partial z_s^0}\frac{\partial z_s^0}{\partial t_s}\frac{\partial t_s}{\partial t}\right\}.$$

The last expression, on account of the equality $\partial t_s/\partial t = 1 - \partial\tau/\partial t$, is transformed to the formula

$$\frac{\partial\tau}{\partial t} = \frac{A_r + B_r - A_s - B_s}{1 - A_s - B_s}, \quad (13)$$

where for brevity of presentation, the intermediate designations

$$A_r = \frac{1}{c_0}\frac{\partial\psi}{\partial x_r^0}\frac{\partial x_r^0}{\partial t},\ B_r = \frac{1}{c_0}\left\{\frac{\partial\psi}{\partial y_r^0}\frac{\partial y_r^0}{\partial t} + \frac{\partial\psi}{\partial z_r^0}\frac{\partial z_r^0}{\partial t}\right\},$$

$$A_s = \frac{1}{c_0}\frac{\partial\psi}{\partial x_s^0}\frac{\partial x_s^0}{\partial t_s},\ B_s = \frac{1}{c_0}\left\{\frac{\partial\psi}{\partial y_s^0}\frac{\partial y_s^0}{\partial t_s} + \frac{\partial\psi}{\partial z_s^0}\frac{\partial z_s^0}{\partial t_s}\right\}. \quad (14)$$

were introduced.

Substituting Eq. (13) into Eq. (7), we derive the preliminary variant of the formula for the Doppler effect in the form

$$\omega_r = \omega_s\,\frac{1 - A_r - B_r}{1 - A_s - B_s}, \quad (15)$$

in which coefficients A_s, B_s, A_r, and B_r should be specified.

The rates of change of the coordinates of points $M_s(x_s^0, y_s^0, z_s^0)$ and $M_r(x_r^0, y_r^0, z_r^0)$ are determined by the velocities of the source $\mathbf{w}$ and receiver $\mathbf{u}$. Therefore, in Eq. (14)

$$\frac{\partial x_s^0(t_s)}{\partial t_s} = \mathbf{w}\cdot\mathbf{e}_x^0(M_s),\quad \frac{\partial y_s^0(t_s)}{\partial t_s} = \mathbf{w}\cdot\mathbf{e}_y^0(M_s),$$

$$\frac{\partial z_s^0(t_s)}{\partial t_s} = \mathbf{w}\cdot\mathbf{e}_z^0(M_s),\quad \frac{\partial x_r^0}{\partial t} = \mathbf{u}\cdot\mathbf{e}_x^0(M_r),$$

$$\frac{\partial y_r^0}{\partial t} = \mathbf{u}\cdot\mathbf{e}_y^0(M_r),\quad \frac{\partial z_r^0}{\partial t} = \mathbf{u}\cdot\mathbf{e}_z^0(M_r), \quad (16)$$

where $(\mathbf{e}_x^0, \mathbf{e}_y^0, \mathbf{e}_z^0)$ is the local orthonormal basis of the coordinate system (x^0, y^0, z^0). We note that in the curvilinear coordinate system the orientation of the local basis may change from point to point. Here, the coordinate system (x^0, y^0, z^0) was introduced so that

$$\mathbf{e}_x^0(M_s) = \mathbf{s}_s \quad \text{and} \quad \mathbf{e}_x^0(M_r) = \mathbf{s}_r, \quad (17)$$

where $\mathbf{s}_s$ and $\mathbf{s}_r$ are the unit vectors tangential to the ray M_sM_r at its end points M_s and M_r (in the refractive medium, as a rule, $\mathbf{s}_s \neq \mathbf{s}_r$).

Because the derivative of the scalar function $f(x, y, z)$ with respect to the direction is equal to the projection of its gradient $\mathrm{grad} f = \nabla f$ onto the given direction, in Eq. (14) we also have

$$\frac{\partial\psi}{\partial x_s^0} = \{\mathbf{e}_x^0 \nabla\psi\}\Big|_{M=M_s} = \mathbf{e}_x^0(M_s)\, \nabla_s\psi,$$

$$\frac{\partial\psi}{\partial y_s^0} = \{\mathbf{e}_y^0 \nabla\psi\}\Big|_{M=M_s} = \mathbf{e}_y^0(M_s)\, \nabla_s\psi,$$

$$\frac{\partial\psi}{\partial z_s^0} = \{\mathbf{e}_z^0 \nabla\psi\}\Big|_{M=M_s} = \mathbf{e}_z^0(M_s)\, \nabla_s\psi,$$

$$\frac{\partial\psi}{\partial x_r^0} = \{\mathbf{e}_x^0 \nabla\psi\}\Big|_{M=M_r} = \mathbf{e}_x^0(M_r)\, \nabla_r\psi,$$

$$\frac{\partial\psi}{\partial y_r^0} = \{\mathbf{e}_y^0 \nabla\psi\}\Big|_{M=M_r} = \mathbf{e}_y^0(M_r)\, \nabla_r\psi,$$

$$\frac{\partial\psi}{\partial z_r^0} = \{\mathbf{e}_z^0 \nabla\psi\}\Big|_{M=M_r} = \mathbf{e}_z^0(M_r)\, \nabla_r\psi, \qquad (18)$$

where $\nabla_s = \left(\frac{\partial}{\partial x_s} + \frac{\partial}{\partial y_s} + \frac{\partial}{\partial z_s}\right)$ and $\nabla_r = \left(\frac{\partial}{\partial x_r} + \frac{\partial}{\partial y_r} + \frac{\partial}{\partial z_r}\right)$ are the operators of differentiation with respect to the points M_s and M_r in the Cartesian coordinate system (x, y, z).

In Eq. (18) we take advantage of the Blokhintsev eikonal equation[7]:

$$|\nabla\psi| = c_0/c\,(1 - \mathbf{v}\nabla\psi/c_0), \qquad (19)$$

which is modified here as follows. Because $\nabla\psi = |\nabla\psi|\mathbf{n}$, Eq. (19) can be reduced to the vector equation

$$\nabla\psi(\mathbf{r}) = \mathbf{n}(\mathbf{r})\, c_0/W(\mathbf{r}), \qquad (20)$$

where $W = c + \mathbf{v}\cdot\mathbf{n}$ is the phase velocity of sound in the coordinate system L. Below we express the vector $\nabla\psi$ in Eq. (20) in terms of two orthogonal components: along the ray and transverse to it. With this purpose, let us examine Fig. 1, where η denotes the angle between the normal $\mathbf{n}$ and the unit vector $\mathbf{s}$, tangential to the ray, and φ denotes the angle between the vectors $\mathbf{v}$ and $\mathbf{s}$. Because the angle between $\mathbf{v}$ and $\mathbf{n}$ is equal to $\varphi + \eta$, we have

$$\mathbf{v}\cdot\mathbf{n} = v\cos(\varphi + \eta) = v_s \cos\eta - v_{\perp s}\sin\eta,$$

where $v_s = v\cos\varphi$ and $v_{\perp s} = v\sin\varphi$ are the longitudinal $(\mathbf{v}_s)$ and transverse $(\mathbf{v}_{\perp s})$ components of the vector $\mathbf{v}$ relative to the ray. Therefore, taking into account the equalities $\cos\eta = W/U$ and $\sin\eta = v_{\perp s}/c$, we obtain

$$\mathbf{n} = \mathbf{s}\, W/U - \mathbf{v}_{\perp s}/c. \qquad (21)$$

Substituting Eq. (21) into Eq. (20), we derive the equation for the eikonal gradient

$$\nabla\psi = \frac{c_0}{U}\mathbf{s} - \frac{c_0}{c}\frac{\mathbf{v}_{\perp s}}{W}, \qquad (22)$$

the right side of which is the difference between two orthogonal vectors.

On account of Eqs. (16)–(18) and (22), formulas (14) assume the form

$$A_r = \mathbf{u}\cdot\mathbf{s}_r/U_r,$$

$$B_r = -\{(\mathbf{u}\cdot\mathbf{e}_y^0(M_r))\,(\mathbf{v}_{\perp s}(M_r)\cdot\mathbf{e}_y^0(M_r)) + $$
$$+ (\mathbf{u}\cdot\mathbf{e}_z^0(M_r))\,(\mathbf{v}_{\perp s}(M_r)\cdot\mathbf{e}_z^0(M_r))\}/(c_r W_r),$$

$$A_s = \mathbf{w}\cdot\mathbf{s}_s/U_s, \qquad (23)$$

$$B_s = -\{(\mathbf{w}\cdot\mathbf{e}_y^0(M_s))\,(\mathbf{v}_{\perp s}(M_s)\cdot\mathbf{e}_y^0(M_s)) + $$
$$+ (\mathbf{w}\cdot\mathbf{e}_z^0(M_s))(\mathbf{v}_{\perp s}(M_s)\cdot\mathbf{e}_z^0(M_s))\}/(c_s W_s),$$

where $W_s = c_s + \mathbf{v}_s\cdot\mathbf{n}_s$; $W_r = c_r + \mathbf{v}_r\cdot\mathbf{n}_r$; $U_s = |c_s\,\mathbf{n}_s + \mathbf{v}_s|$; $U_r = |c_r\,\mathbf{n}_r + \mathbf{v}_r|$; $\mathbf{n}_s = \mathbf{n}(M_s)$; $\mathbf{n}_r = \mathbf{n}(M_r)$; $c_s = c(M_s)$; $c_r = c(M_r)$; $\mathbf{v}_s = \mathbf{v}(M_s)$; $\mathbf{v}_r = \mathbf{v}(M_r)$ are the values of the parameters of the sound wave and the medium at the end points of the ray M_sM_r.

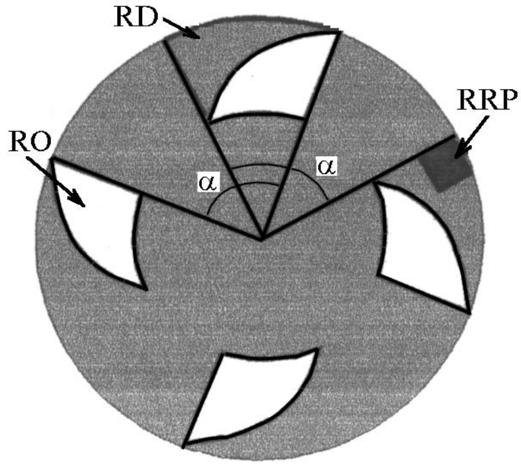

Fig 1. Geometric relationships of the phase (**W**) and group (**U**) velocities of sound with the normal to the wavefront **n** and the unit vector **s**, tangential to the ray, in the medium moving with the velocity **v**.

The expressions for B_r and B_s in Eq. (23) are affixed to the orientation of the axes of the curvilinear coordinate system (x^0, y^0, z^0) transverse to the ray. At the same time, this coordinate system was introduced in such a manner that the orientation of its unit vectors $\mathbf{e}_y^0$ and $\mathbf{e}_z^0$ in the plane orthogonal to $\mathbf{e}_x^0$ was unimportant for the problem at hand and can be arbitrary. For vivid presentation of final results, it is most convenient to specify it in the above-indicated plane relative to the direction of physical significance. Let, for example, the unit vector $\mathbf{e}_y^0$ at the point M_r be directed along the vector $\mathbf{v}_{\perp s}(M_r)$, that is, $\mathbf{e}_y^0(M_s) = \mathbf{v}_{\perp s}(M_s)/v_{\perp s}(M_s)$. Then the expression for B_r from Eq. (23), by virtue of orthogonality of $\mathbf{e}_y^0$ and $\mathbf{e}_z^0$, is transformed to the formula

$$B_r = -\{\mathbf{u}\cdot\mathbf{v}_{\perp s}(M_r)\}/(c_r W_r).$$

If we denote by $\mathbf{u}_{\perp s} = \mathbf{u} - (\mathbf{u}\cdot\mathbf{s}_r)\cdot\mathbf{s}_r$ the component of the receiver velocity $\mathbf{u}$, transverse to the direction of sound propagation, in view of the identity $\mathbf{u}\cdot\mathbf{v}_{\perp s}(M_r) \equiv \mathbf{u}_{\perp s}\cdot\mathbf{v}(M_r)$, the last expression for B_r also can be written as

$$B_r = -\,\mathbf{u}_{\perp s}\cdot\mathbf{v}_r/(c_r\,W_r). \qquad (24)$$

In analogy with Eq. (24), the expression for B_s has the form

$$B_s = -\,\mathbf{w}_{\perp s}\cdot\mathbf{v}_s/(c_r\,W_r), \qquad (25)$$

where $\mathbf{w}_{\perp s} = \mathbf{w} - (\mathbf{w}\cdot\mathbf{s}_s)\cdot\mathbf{s}_s$ is the component of the source velocity $\mathbf{w}$ transverse to the direction of sound propagation.

Substituting Eqs. (23)–(25) into Eq. (15), we obtain

$$\omega_r = \omega_s\,\frac{1 - \mathbf{u}\cdot\mathbf{s}_r/U_r + \mathbf{u}_{\perp s}\cdot\mathbf{v}_r/(c_r\,W_r)}{1 - \mathbf{w}\cdot\mathbf{s}_s/U_s + \mathbf{w}_{\perp s}\cdot\mathbf{v}_s/(c_s\,W_s)}\,. \qquad (26)$$

Formula (26) is the final formula for the acoustic Doppler effect in the three-dimensional inhomogeneous moving medium, in which the velocities of the source and receiver are considered relative to the physical direction of wave propagation specified by the vectors $\mathbf{s}_s$ and $\mathbf{s}_r$.

The other variant of the Doppler formulas, alternate to (24), is also known, in which the above-indicated velocities are considered relative to the normal to the wavefront $\mathbf{n} = \nabla\psi/|\nabla\psi|$ (or relative to the wave vector $\mathbf{K} = k_0\nabla\psi$, where $k_0 = \omega/c_0$). In particular, classical formula (1) for the acoustic Doppler effect in the homogeneous stationary medium has this form. To reduce formula (26) to Eq. (1), we take advantage of the equalities $\mathbf{w}_{\perp s}\cdot\mathbf{v}(M_s) \equiv \mathbf{w}\cdot\mathbf{v}_{\perp s}(M_s)$ and $\mathbf{u}_{\perp s}\cdot\mathbf{v}(M_r) \equiv \mathbf{u}\cdot\mathbf{v}_{\perp s}(M_r)$. Substituting them into Eq. (26), on account of Eq. (21) we obtain

$$\omega_r = \omega_s\,\frac{1 - \mathbf{n}_r\cdot\mathbf{u}/W_r}{1 - \mathbf{n}_s\cdot\mathbf{w}/W_s}\,. \qquad (27)$$

The choice of the reference (conventionally stationary) system of coordinates L, in which the motion of the source and receiver is considered, generally speaking, is arbitrary and is dictated only by the convenience of physical interpretation of the Doppler effect. For example, in acoustics of the atmosphere and ocean the system of coordinates $\hat{L}$, affixed to a certain point of the Earth's surface, is considered stationary. One more widespread variant of the stationary system of coordinates is affixed to the receiver (in this case, $L = L''$). The reason of the last choice is that the observer, who measures the sound frequency, is usually near the receiver and, as a rule, moves with it (for example, on board an aircraft or a ship). In this case, in analogy with the SRT (Refs. 1–3), it is convenient to introduce the notion of the relative velocity of the source and receiver $\boldsymbol{\upsilon} = \mathbf{w} - \mathbf{u}$, equal to the source velocity $\mathbf{w}''$ in the coordinate system L''.

It is not difficult to transform formula (27) at the transition from one coordinate system L to the other $\hat{L}$, moving relative to the first system with a constant velocity $\mathbf{m}$ (the condition of inertial systems). The rule of addition of velocities, following from the Galilean transformations, has the form

$$\mathbf{v} = \hat{\mathbf{v}} + \mathbf{m},$$

where $\mathbf{v}$ and $\hat{\mathbf{v}}$ are the velocities of a material point in coordinate systems L and $\hat{L}$. It is also known[1] that from the invariance of the wave phase, the invariance of the normal to the wavefront follows (in this case, this is true for $\mathbf{n}_s$ and $\mathbf{n}_r$). Therefore, the Doppler effect in the new coordinate system $\hat{L}$ is described by the formula formally identical to Eq. (27)

$$\omega_r = \omega_s\,\frac{1 - \mathbf{n}_r\cdot\hat{\mathbf{u}}/\hat{W}_r}{1 - \mathbf{n}_s\cdot\hat{\mathbf{w}}/\hat{W}_s}\,,$$

where $\hat{\mathbf{u}} = \mathbf{u} - \mathbf{m}$; $\hat{\mathbf{w}} = \mathbf{w} - \mathbf{m}$; $\hat{W}_s = c_s + \hat{\mathbf{v}}_s\cdot\mathbf{n}_s$; $\hat{W}_r = c_r + \hat{\mathbf{v}}_r\cdot\mathbf{n}_r$; $\hat{\mathbf{v}}_s = \mathbf{v}_s - \mathbf{m}$ and $\hat{\mathbf{v}}_r = \mathbf{v}_r - \mathbf{m}$. In the coordinate system L'', affixed to the receiver, the Doppler formula has the form $\omega_r = \omega_s/(1 - \mathbf{n}_s\boldsymbol{\upsilon}/W''_s)$, analogous to the formula which follows from Eq. (27) in case of the moving sound source only.

The simplicity of these final formulas for the acoustic Doppler effect in the inhomogeneous moving medium is apparent. To estimate the Doppler frequency shift from these formulas, the refraction problem on the orientation of the vectors $\mathbf{s}_s$ and $\mathbf{s}_r$ (or $\mathbf{n}_s$ and $\mathbf{n}_r$) should be additionally solved. This class of problems can be solved analytically only for a stratified medium with the use, as a rule, approximate methods (see, for example, Refs. 9 and 18).

In many cases it is required to reconstruct numerically the trajectory of the sound ray from the source to the receiver by its computer calculation for the given spatial distributions of $c(\mathbf{r})$ and $\mathbf{v}(\mathbf{r})$ (see Refs. 19 and 20).

Analysis of the results

First of all we dwell on the physical meaning of formula (27) by its comparison with formula (1). In classical formula (1), describing the Doppler effect in the homogeneous stationary medium, the adiabatic sound velocity c is simultaneously its phase velocity ($W = c$), and the vector $\mathbf{n}$ coincides with the normals to the wavefront at the points of sound emission ($\mathbf{n} = \mathbf{n}_s$) and reception ($\mathbf{n} = \mathbf{n}_r$). Therefore, based on Eq. (1), we conclude that the acoustic Doppler effect in the homogeneous stationary medium depends on the ratios of the projections of the source and receiver velocities onto the normal to the wavefront and to the phase velocity of wave propagation. In its turn, formula (27) demonstrates that this physical pattern is also observed for the inhomogeneous moving medium; moreover, because the values of these ratios differ at different points, they should be taken into account only for the end points of the ray connecting the source and the receiver.

Formula (27) also can be used to study the Doppler effect when the source and the receiver are completely entrained by the nonuniform movement of the medium, that is, when they are stationary relative to the medium, but move relative to each other. In this case, $\mathbf{w} = \mathbf{v}_s$ and $\mathbf{u} = \mathbf{v}_r$, and on account of the formula for the phase sound velocity, Eq. (27) is reduced to the form

$$\omega_r = \omega_s \frac{1 + \mathbf{n}_s \cdot \mathbf{v}_s / c_s}{1 + \mathbf{n}_r \cdot \mathbf{v}_r / c_r}. \quad (28)$$

Because in an inhomogeneous moving medium, as a rule, $\mathbf{n}_s \mathbf{v}_s / c_s \neq \mathbf{n}_r \mathbf{v}_r / c_r$, from Eq. (28) it follows that the frequency shift of oscillations recorded by the receiver, in this case also should be observed.

If we recall formula (2) for the optical Doppler effect in vacuum, we point out that when the direction of the observed optical ray is perpendicular to the velocity υ (for $\upsilon \cdot \mathbf{s} = 0$), the transverse Doppler effect occurs, described by the formula

$$\omega_r = \omega_s \sqrt{1 - \upsilon^2 / c_e^2}. \quad (29)$$

This effect is manifested for the electromagnetic waves as a consequence of the SRT, namely, because of the unequal passage of times in the coordinate systems moving relative to each other. Therefore, its occurrence is usually indicated as a distinctive feature of the Doppler effect for the electromagnetic waves in comparison with the acoustic ones.

From above-derived formula (26) it follows that in acoustics of the moving media the frequency shift of the received oscillations also can be observed when the wave propagates perpendicularly to the direction of motion of the source or the receiver, that is, when $\mathbf{w} \cdot \mathbf{s} = 0$ or $\mathbf{u} \cdot \mathbf{s} = 0$. The shift magnitude is given by the formula

$$\omega_r = \omega_s \frac{1 + \mathbf{u}_{\perp s} \cdot \mathbf{v}_r / (c_r\ W_r)}{1 + \mathbf{w}_{\perp s} \cdot \mathbf{v}_s / (c_s\ W_s)}$$

$$\text{or} \quad \omega_r = \omega_s / \{1 + \upsilon\ \mathbf{v}_s'' / (c_s\ W_s'')\},$$

where $\mathbf{v}_s'' = \mathbf{v}_s - \mathbf{v}_r$, which differs from Eq. (29). Because this phenomenon formally satisfies the definition of the transverse Doppler effect, it makes sense to use the same name for it.

According to Eq. (26), the motion of the source or the receiver causes the change of time τ of sound propagation between them primarily due to the change of the geometric path length S along the ray trajectory connecting them, and by virtue of Eq. (7) causes the longitudinal Doppler effect. In this case, the contribution of the changes ΔU of the group sound velocity U along its propagation path to the Doppler frequency shift has considerably less effect because of small values of $\Delta U / U$ for realistic media. When the source or the receiver moves perpendicular to the direction of wave propagation, the sound wave arrives at the point of reception at each moment after passage along the new ray with different emission angles. These rays can be considered as one nonstationary ray connecting the source and the receiver, which bends depending on the instantaneous position of the source or the receiver. Because in the moving (anisotropic for sound waves) medium the group velocity of sound depends on the direction of wave propagation, ray bending will cause additional change of τ even for the fixed path length S. The last phenomenon described by Eq. (26) is referred to as the transverse Doppler effect.

It should be noted that for the unit vector $\mathbf{s}$ that specifies the direction of sound wave propagation, the following formula is valid[13]

$$\mathbf{s} = (\mathbf{n} + \mathbf{v}/c)/(1 + 2\mathbf{v}\mathbf{n} + \mathbf{v}^2/c^2)^{1/2}.$$

The normal to the wavefront $\mathbf{n}$, as already mentioned, is invariant. At the same time, in case of transition from one inertial system of coordinates to the other, moving relative to the first system, the velocity of the medium $\mathbf{v}$ changes. In this case, in accordance with the last formula, the direction of $\mathbf{s}$ also changes, i.e., the direction of sound wave propagation is not invariant. Therefore, the magnitude of the Doppler frequency shift caused by the transverse Doppler effect depends on the choice of the coordinate system.

In case of the homogeneous moving medium ($c(\mathbf{r}) = \text{const}$ and $\mathbf{v}(\mathbf{r}) = \text{const}$), $\mathbf{s}_r = \mathbf{s}_s = \mathbf{s}$, and Eq. (26) reduces to

$$\omega_r = \omega_s \frac{1 - \mathbf{u} \cdot \mathbf{s} / U + \mathbf{u}_{\perp s} \cdot \mathbf{v} / (c\ W)}{1 - \mathbf{w} \cdot \mathbf{s} / U + \mathbf{w}_{\perp s} \cdot \mathbf{v} / (c\ W)}, \quad (30)$$

where $U = |c\mathbf{n} + \mathbf{v}|$ and $W = c + \mathbf{v} \cdot \mathbf{n}$. Formula (30) demonstrates that the transverse Doppler effect can be observed even in the absence of refraction, when in the coordinate system L, in which this phenomenon is studied, the medium moves. However, it is easy to notice that unlike the general case $c(\mathbf{r})$ and $\mathbf{v}(\mathbf{r}) \neq \text{const}$, for the homogeneous moving medium we always can choose the absolute system of coordinates, in which $\mathbf{v} \equiv 0$ and hence there is no transverse Doppler effect. In the absolute system of coordinates, Eq. (30) has the form

$$\omega_r = \omega_s \frac{1 - \mathbf{u} \cdot \mathbf{s} / U}{1 - \mathbf{w} \cdot \mathbf{s} / U}.$$

Because in the stationary medium $\mathbf{s} = \mathbf{n}$ and $U = c$, the last formula is identical to Eq. (1).

Known formulas for the Doppler effect in acoustics

The derivation of classical formula (1) for the acoustic Doppler effect in the homogeneous stationary medium, also starting from the invariance of the wave phase, was described, for example, in Refs. 1 and 2. Analogous formula was reported in Ref. 4 without derivation. At the same time, in Refs. 3 and 6 the formula analogous to Eq. (1), comprised, instead of $\mathbf{w} \cdot \mathbf{n}$, the projection of $\mathbf{w}$ onto the direction $\mathbf{p} = \{\mathbf{r}_r(t) - \mathbf{r}_s(t)\} / |\mathbf{r}_r(t) - \mathbf{r}_s(t)|$, along which the source is seen at the moment of reception of sound waves t. Because for the moving source the vectors $\mathbf{n}$ and $\mathbf{p}$ do not coincide, a mistake was introduced in Refs. 3 and 6, second-order infinitesimal in w/c. The exact formula for the Doppler effect in the homogeneous medium in terms of the projections of $\mathbf{u}$ and $\mathbf{w}$ on the direction $\mathbf{p}$ was derived by Blokhintsev.[7] It comprised the quadratic term w^2/c^2. The formula for the frequency, derived in Ref. 21, also contained w^2/c^2. However, because in this work the relative position of the source and the receiver was characterized by the normal **n**, its results are incorrect.

In the system of coordinates, in which the homogeneous medium moves with the velocity **v**, the velocities of the source and receiver are equal to **w** – **v** and **u** – **v**, respectively. Substituting them in Eq. (1) and taking into account the invariance of **n**, we obtain

$$\omega_r = \omega_s \frac{1 + \mathbf{u}\cdot(\mathbf{v} - \mathbf{u})/c}{1 + \mathbf{w}\cdot(\mathbf{v} - \mathbf{w})/c} .$$

This formula was given by Ostashev[13] as the formula for the Doppler effect in the homogeneous moving medium.

The acoustic Doppler effect in the inhomogeneous moving medium was considered for the first time in Ref. 8. The work was aimed at an analysis of errors in measuring the wind velocities with Doppler sodars (acoustic radars). However, several principal errors were committed in it. In particular, in Ref. 8 the length of the ray trajectory connecting the source and the receiver and hence the increment of the eikonal $\psi(\mathbf{r}_s, \mathbf{r}_r)$ depended on the coordinates of the source at the moment of sound reception t, rather than at the moment of sound emission t_s. This resulted in disappearance of the nonlinearity in the Doppler formula with respect to the velocity of the source **w** (or υ), well known in acoustics. In addition, Ref. 8 ignored ray bending with time, caused by the transverse component of the velocity of the source or receiver. Because of this, the contribution of the above-described transverse Doppler effect to the Doppler frequency shift $\omega_d = \omega_r - \omega_s$ was not taken into account. It is not difficult to notice that the resulting relative error in estimating ω_d in Ref. 8 was compared by the order of magnitude with $\varepsilon = \max\{|\mathbf{v}|/c_0 \ll 1, |c - c_0|/c_0 \ll 1\}$. Therefore, from the results of this work the contribution of refraction to ω_r cannot be estimated even with the accuracy of the linear approximation in ε.

Our desire to correct the errors committed in Ref. 8 and to obtain the correct formulas for estimation of the influence of the atmospheric stratification on the operation of Doppler sodars considering the terms linear in ε has resulted in appearance of Refs. 9 and 10. Because the results obtained in Refs. 9 and 10 were of independent interest, later they were generalized in Ref. 11 for the three-dimensional inhomogeneous moving medium, where the transverse Doppler effect in acoustics of the moving media was pointed out for the first time.

Simultaneously with Ref. 11, the Doppler effect in the three-dimensional inhomogeneous moving medium was also studied by Ostashev,[12] who suggested the formula

$$\omega_r = \omega_s \frac{1 + \mathbf{s}_s\cdot\mathbf{v}_s/c_s}{1 + \mathbf{s}_s\cdot(\mathbf{v}_s - \mathbf{w})/c_s} \frac{1 + \mathbf{s}_r\cdot(\mathbf{v}_r - \mathbf{u})/c_r}{1 + \mathbf{s}_r\cdot\mathbf{v}_r/c_r} , \qquad (31)$$

where $\mathbf{s}_s$ and $\mathbf{s}_r$ are the unit vectors directed from the source to the receiver in the coordinate systems moving with the velocities $\mathbf{v}_s$ and $\mathbf{v}_r$, respectively.

Formula (31) was derived in Ref. 12 based on physical reasoning, the essence of which can be briefly expressed as follows. The medium was examined, in which the characteristic scale a of variations of the sound velocity c and of the velocity of the medium **v** was much greater than the wavelength λ and the sizes of the source or receiver d. In the vicinity of the source and receiver, moving with constant velocities **w** and **u**, the regions D_s and D_r were separated, whose dimensions were much greater than λ and d, but less than a. In these regions c and **v** were considered constant and equal to c_s, $\mathbf{v}_s$, c_r, and $\mathbf{v}_r$, respectively. The Doppler effect was studied only within the regions D_s and D_r on the basis of classical formula (1), using the rule of addition of velocities at transition from one inertial system of coordinates to the other. In the Doppler formulas, derived for the regions D_s and D_r, the directions of vectors $\mathbf{s}_s$ and $\mathbf{s}_r$ were not specified, because they were considered interrelated by the refraction laws. In case of simultaneous motion of the source and receiver, the formulas for the regions D_s and D_r were combined on the basis of the assumption that, when the wave propagated in the inhomogeneous moving medium, its frequency in the chosen (conventionally stationary) coordinate system L remained unchanged. As a result, formula (31) was derived. Later (see, for example, Ref. 13) the vectors $\mathbf{s}_s$ and $\mathbf{s}_r$, entering formula (31), were replaced by the normals $\mathbf{n}_s$ and $\mathbf{n}_r$ coinciding with them.

Formula (31) and analogous expression derived in Ref. 11, significantly differ. Because in Refs. 11 and 12 different approaches to the solution of the problem were used, directly from these works it was not clear, how one of these formulas can be transformed into another. Therefore, in Ref. 14 formula (27) for the acoustic Doppler effect in the inhomogeneous moving medium was derived for the first time, based on the approach used in Refs. 9–11. If in the last formula we express the phase velocities of sound W_s and W_r in terms of c and **v**, it is easily transformed into Eq. (31). Formula (26) derived here was not published previously in the literature.

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