OPTIMAL ALGORITHM FOR ESTIMATING THE STATES OF THE MC-EVENT FLUX IN THE PRESENCE OF ERRORS IN MEASUREMENTS OF TIME

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The MC (Markovian chain) flux of events is a mathematical model of the fluxes of elementary particles (photons, electrons, etc.). To estimate its state, we have derived a recursion formula for the a posteriori probability that is most comprehensive characteristic of the state of an event flux. The decision on the flux state is made using a criterion of the a posteriori probability maximum. We present here some results of calculations of the a posteriori probability along with the results of statistical experiment using an imitation model.

1. INTRODUCTION

Random flux of events is a widely used, in mathematical simulation, model of real processes. Thus, the fluxes of elementary particles (photons, electrons, and so on, coming to the measurement devices), information circulating in the computer and communication networks may adequately be described as a random flux of events. The problems on estimating the state and parameters of a random flux of events arises in optical and laser systems operating in the photon counting mode (for instance, in laser sensing of the upper layers of the atmosphere), in optical detection, recognition, and tracking systems operating through the atmosphere at ultimately long distances, as well as in optical systems of communication beyond the horizon.

The majority of authors consider mathematical models of the flux of events under the assumption that the event occurrence in time is determined without errors. However, the devices detecting the events add introduce errors into the measurements. These errors must be taken into account when making a decision based on statistical data.

In Ref. 1 one may found analysis of an empirical algorithm for estimating the states of MC-flux of events. The algorithm is based on consideration of a weighting function of observations that allows for aging of the observations. In this paper, we derive a recursion relation for *a posteriori* probability for the states of MC-flux of events assuming the occurrence of an event to be determined with an error. The decision on the state of a flux is made using the criterion of *a posteriori* probability maximum. The *a posteriori* probability is the most comprehensive characteristic of a flux state among those that may be obtained from only an observation sample available. In this case the *a posteriori* probability gives minimum to full probability of making a false decision.[2]

2. STATEMENT OF THE PROBLEM

We consider a doubly stochastic Poisson flux of events with its intensity being a piece-wise constant stationary random process $\lambda(t)$ having two states, λ_1 and λ_2 ($\lambda_1 > \lambda_2$). We shall say that the process is in its first state if $\lambda(t) = \lambda_1$ and, otherwise, in the second one when $\lambda(t) = \lambda_2$. Durations of the process staying in one or the other state are distributed according to the exponential law $F_i(t) = 1-\exp\{-\alpha_i\, t\}$, $i = 1,2$, where α_1 is the rate of change of the first state for the second one; α_2 is the rate of change of the second state for the first one. Within the stationary intervals (when $\lambda(t) = \lambda_1$ or $\lambda(t) = \lambda_2$), the flux of events observed is of Poisson statistics. This flux is called the MC-flux or the switching process. Since the process $\lambda(t)$ is not observed and only the occurrence of events is observed, it is necessary to estimate the state of a flux from observations at a given moment. The measurements of the time of an event occurrence contain an error $t_i = t_i^0 + \Delta t_i$, where t_i are observed times of the events' occurrence; t_i^0 are true times of the events' occurrence; Δt_i are measurement errors which are independent and similarly distributed for all i. Let us assume that the measurement error is normal, has zero mean value and the variance σ^2. Since measurement errors inevitably lead to the confusion of the events (that means that the event occurring at the moment t_i^0 can be observed at the moment $t_i < t_i^0$ or $t_i > t_i^0$), let us take that $\sigma \ll 1$ (this indicates that the detecting devices are not so bad to change them for others). Thereby, the case of total confusion of the events is avoided with a high probability (confusion is possible only for neighboring events).

We consider the stationary flux of events, so we neglect the transition processes during the observation interval (t_0, t), where t_0 is the beginning of the observations, t is the end of the observations (the

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moment for making a decision). Then, without any loss of generality, one can assume that $t_0 = 0$ and $0 < t_i = t_i^0 + \Delta t_i < t$. Note that the *a priori* probabilities of the first and second states of the process at the time moment t are determined in the form

$$\pi_1(t,\ t_0) = \frac{\alpha_2}{\alpha_1 + \alpha_2} + \left(q - \frac{\alpha_2}{\alpha_1 + \alpha_2}\right) \times$$
$$\times \exp\{-(\alpha_1 + \alpha_2)(t - t_0)\};$$

$$\pi_2(t,\ t_0) = \frac{\alpha_1}{\alpha_1 + \alpha_2} - \left(q - \frac{\alpha_2}{\alpha_1 + \alpha_2}\right) \times$$
$$\times \exp\{-(\alpha_1 + \alpha_2)(t - t_0)\},$$

where q is the probability that the first state of the process $\lambda(t)$ takes place at the moment t_0. Then, for the stationary regime $(t \to \infty)$, we obtain the final *a posteriori* probabilities of the states in the form[3]

$$\pi_1 = \frac{\alpha_2}{\alpha_1 + \alpha_2}, \quad \pi_2 = \frac{\alpha_1}{\alpha_1 + \alpha_2}.$$

Thus, to estimate the state of the unobserved process $\lambda(t)$ at a time moment t, it is necessary to determine the *a posteriori* probabilities $w(\lambda_1/t_1, \ldots, t_n)$ and $w(\lambda_2/t_1, \ldots, t_n)$ that, in the time moment t, $\lambda(t) = \lambda_1$ or $\lambda(t) = \lambda_2$, respectively, (n is the number of observations during the time t). Because it is valid that $w(\lambda_1/t_1, \ldots, t_n) + w(\lambda_2/t_1, \ldots, t_n) = 1$, it is sufficient to determine only one *a posteriori* probability, for example, $w(\lambda_1/t_1, \ldots, t_n)$. The decision on the state of a process is made by comparing the *a posteriori* probabilities: if $w(\lambda_1/t_1, \ldots, t_n) \geq w(\lambda_2/t_1, \ldots, t_n)$, then $\lambda(t) = \lambda_1$; otherwise, $\lambda(t) = \lambda_2$. Finally, let us note that the problem of estimating states of an MC-flux of events was solved in Ref. 3 under the assumption that there are no errors in measurements of the occurrence time.

3. DERIVATION OF THE TRANSITION PROBABILITIES

Let the time vary discretely with a finite step Δt: $t = k\Delta t$, $k = 0, 1, \ldots$. Let us consider a two-dimensional process $(\lambda^{(k)}, r_k)$, where $\lambda^{(k)} = \lambda(k\Delta t)$ is the value of the process $\lambda(t)$ at the moment $k\Delta t$ ($\lambda^{(k)} = \lambda_1$ or $\lambda^{(k)} = \lambda_2$), $r_k = r_k(\Delta t) = r[k\Delta t] - r[(k-1)\Delta t]$ is the number of events observed within the time interval $((k-1)\Delta t, k\Delta t)$ of the length Δt, $r_k = 0, 1, \ldots$.

Let us consider the probability $p(\lambda^{(k+1)}, r_{k+1}/\lambda^{(k)}, r_k)$ which is the conditional probability that the process $(\lambda^{(k)}, r_k)$ being in the state $(\lambda^{(k)}, r_k)$, at the moment $k\Delta t$, will take the state $(\lambda^{(k+1)}, r_{k+1})$ at the moment $(k + 1)\Delta t$. In other words, $p(\lambda^{(k+1)}, r_{k+1}/\lambda^{(k)}, r_k)$ is the probability of transition of the process $(\lambda^{(k)}, r_k)$ from one state to the other in one step Δt. Then

$$p(\lambda^{(k+1)}, r_{k+1}/\lambda^{(k)}, r_k) = p(\lambda^{(k+1)}/\lambda^{(k)}, r_k) \times$$
$$\times p(r_{k+1}/\lambda^{(k)}, \lambda^{(k+1)}, r_k). \qquad (1)$$

The first factor in Eq. (1) is written as $p(\lambda^{(k+1)}/\lambda^{(k)}, r_k) = p(\lambda^{(k+1)}/\lambda^{(k)})$ because the number of events r_k observed within the interval $((k-1)\Delta t, k\Delta t)$ does not influence the value of the process $\lambda(t)$ at the moment $(k+1)\Delta t$ (the process $\lambda(t)$ «lives its own life[ ); as to the value $\lambda^{(k)}$ of the process $\lambda(t)$ at the moment $k\Delta t$, it does not depend on the prehistory because of the exponential distribution of duration of the state $\lambda^{(k)}$; finally, the measurement errors do no effect the state of the process $\lambda(t)$.

Now let us consider the second factor in Eq. (1) where r_{k+1} is the number of events observed within the interval $(k\Delta t, (k+1)\Delta t)$. Since, generally speaking, the measurement errors may result in mixing the events along the entire temporal axis the probability $p(r_{k+1}/\lambda^{(k)}, \lambda^{(k+1)}, r_k)$ can be written in the following form:

$$p(r_{k+1}/\lambda^{(k)}, \lambda^{(k+1)}, r_k) =$$
$$= p(r_{k+1}/\lambda^{(k)}, \lambda^{(k+1)}, r_k(\ldots, \lambda^{(0)}, \lambda^{(1)}, \ldots, \lambda^{(k)}, \ldots)). \qquad (2)$$

Let us choose $\sigma \ll 1$ such that the probability of transitions between the events from intervals not adjacent to the interval $(k\Delta t, (k+1)\Delta t)$ is sufficiently small and we neglect those seldom transitions, i.e., an event occurring at the moment $t_i^0 \in ((k-1)\Delta t, k\Delta t)$ or $t_i^0 \in ((k+1)\Delta t, (k+2)\Delta t)$ can be observed only in the interval $(k\Delta t, (k+1)\Delta t)$; and the event occurring at the moment $t_i^0 \in (k\Delta t, (k+1)\Delta t)$ can be observed only either in the interval $((k-1)\Delta t, k\Delta t)$ or in the interval $((k+1)\Delta t, (k+2)\Delta t)$. By virtue of these reasoning, Eq. (2) takes the form

$$p(r_{k+1}/\lambda^{(k)}, \lambda^{(k+1)}, r_k) =$$
$$= p(r_{k+1}/\lambda^{(k-1)}, \lambda^{(k)}, \lambda^{(k+1)}). \qquad (3)$$

Thus, the transition probability (1) takes the form

$$p(\lambda^{(k+1)}, r_{k+1}/\lambda^{(k)}, r_k) =$$
$$= p(\lambda^{(k+1)}/\lambda^{(k)})\ p(r_{k+1}/\lambda^{(k-1)}, \lambda^{(k)}, \lambda^{(k+1)}). \qquad (4)$$

Let us first obtain expressions for the probabilities $p(\lambda^{(k+1)}/\lambda^{(k)})$. To simplify designations, let us make substitutions $k\Delta t = u$, $(k+1)\Delta t = \tau$. Let us consider two adjacent intervals: (u, τ), $u < \tau$, and $(\tau, \tau + \Delta\tau)$, where $\Delta\tau$ is an infinitesimal time interval. Then, it is easy to write the differential equation for the probabilities $p(\lambda(\tau) = \lambda^{(k+1)}/\lambda(u) = \lambda^{(k)})$:

$p'(\lambda(\tau) = \lambda_1/\lambda(u) = \lambda^{(k)}) =$

$= \alpha_2 - (\alpha_1 + \alpha_2)\, p(\lambda(\tau) = \lambda_1/\lambda(u) = \lambda^{(k)}),$

$p(\lambda(\tau) = \lambda_2/\lambda(u) = \lambda^{(k)}) =$

$= 1 - p(\lambda(\tau) = \lambda_1/\lambda(u) = \lambda^{(k)}),\ \lambda^{(k)} = \lambda_1, \lambda_2,$

the solution to which, with the allowance for the designations used, can be written in the form

$$p(\lambda^{(k+1)} = \lambda_1/\lambda^{(k)} = \lambda_1) = \pi_1 + \pi_2 \exp\{-(\alpha_1 + \alpha_2)\,\Delta t\},$$
$$p(\lambda^{(k+1)} = \lambda_2/\lambda^{(k)} = \lambda_1) = \pi_2 - \pi_2 \exp\{-(\alpha_1 + \alpha_2)\,\Delta t\},$$
$$p(\lambda^{(k+1)} = \lambda_1/\lambda^{(k)} = \lambda_2) = \pi_1 - \pi_1 \exp\{-(\alpha_1 + \alpha_2)\,\Delta t\},$$
$$p(\lambda^{(k+1)} = \lambda_2/\lambda^{(k)} = \lambda_2) = \pi_2 + \pi_1 \exp\{-(\alpha_1 + \alpha_2)\,\Delta t\},$$

where π_1, π_2 are defined above. Thus the transition probabilities $p(\lambda^{(k+1)}/\lambda^{(k)})$ in Eq. (4) are determined by the formulas presented above.

Now let us determine the probability $p(r_{k+1}/\lambda^{(k-1)}, \lambda^{(k)}, \lambda^{(k+1)})$. First of all, we state the problem on finding the probability distribution for the number of transitions through the right boundary $(k + 1)\Delta t$ of the interval $(k\Delta t, (k + 1)\Delta t)$ for the events occurring at the moments $t_i^0 \in (k\Delta t, (k + 1)\Delta t)$. Let us denote the numbers of events occurring at the moments t_i^0 in the interval $(k\Delta t, (k + 1)\Delta t)$, as it is shown in Fig. 1. In this Figure, the intervals between neighbor events are designated as $(\tau_i = t_{i-1}^0 - t_i^0,\ i = 2, 3, \ldots,$ $\tau_1 = (k + 1)\Delta t - t_1^0)$. Since Δt is small, the value $\lambda(t) = \lambda^{(k)}$ $(\lambda^{(k)} = \lambda_1$ or $\lambda^{(k)} = \lambda_2)$ is supposed to be constant for $t \in [k\Delta t, (k + 1)\Delta t]$.

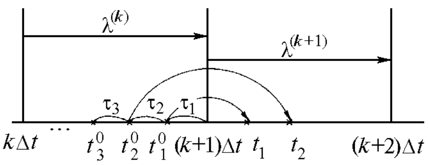

FIG. 1. Illustration of event transitions through the right boundary of the interval $(k\Delta t, (k + 1)\Delta t)$.

Let us consider the time moment t_1^0 and fix the value τ_1. Then, because of measurement errors, the first event can cross the right boundary $(k + 1)\Delta t$ of the interval thus being observed in the adjacent interval (see Fig. 1). This takes place if $t_1 > t_1^0 + \tau_1$. Then, at a fixed τ_1, the probability of this event can be written in the form

$$P_1(\tau^{(1)} = \tau_1) = P(t_1 > t_1^0 + \tau_1) =$$

$$= \frac{1}{\sqrt{2\pi}\,\sigma} \int\limits_{t_1^0+\tau_1}^{+\infty} \exp\left\{-\frac{(t_1 - t_1^0)^2}{2\sigma^2}\right\} \mathrm{d}t_1 = t\ \left(-\frac{\tau^{(1)}}{\sigma}\right).$$

Here and below $t\ (=) = \dfrac{1}{\sqrt{2\pi}} \displaystyle\int\limits_{-\infty}^{a} \exp\left\{-\frac{x^2}{2}\right\} \mathrm{d}x.$

Similarly, the second event can cross the right boundary $(k + 1)\Delta t$ of the interval for the fixed τ_1 and τ_2, if $t_2 > t_2^0 + \tau_1 + \tau_2$. Then

$$P_2(\tau^{(2)} = \tau_1 + \tau_2) = P(t_2 > t_2^0 + \tau_1 + \tau_2) = t\ \ (-\tau^{(2)}/\sigma).$$

Finally, the jth event can cross the right boundary $(k + 1)\Delta t$ of the interval for the fixed $\tau_1, \ldots, \tau_j$, if $t_j > t_j^0 + \tau_1 + \ldots + \tau_j$. Then

$$P_j(\tau^{(j)} = \tau_1 + \ldots + \tau_j) = P(t_j > t_j^0 + \tau_1 + \ldots + \tau_j) =$$
$$= t\ \ (-\tau^{(j)}/\sigma), \quad j = 1, 2, \ldots . \tag{5}$$

Let us consider a random value n_j that may take two values, 0 and 1. When crossing the boundary $(k + 1)\Delta t$, $n_j = 1$, otherwise, $n_j = 0$. Here n_j takes the value 1 with the probability $P_j(\tau^{(j)})$, and value 0 with the probability $1 - P_j(\tau^{(j)})$. Then the probability distribution for the number of events' transitions through the right boundary $(k + 1)\Delta t$ is nothing but the distribution of the random value $n = \sum\limits_{j=1}^{\infty} n_j$. Note that conventional limit theorems for sums of random values cannot be applied here because the probabilities $P_j(\tau^{(j)})$ depend on j. Let us obtain the characteristic function of the random value n because it uniquely determines the probability distribution.[4]

By definition of the characteristic function,[4] and because the random values n_j are independent, we have

$$g(x)\, M\,\{\exp(ixn)\} = M\left\{\prod_{j=1}^{\infty} \exp\,(ixn_j)\right\} = M\left\{\prod_{j=1}^{\infty} z^{n_j}\right\}, \tag{6}$$

where $i = \sqrt{-1}$; $z = \exp(ix)$, x is the real number. First, we average Eq. (6) over the numbers of 0 and 1 states of the random value n_j at fixed $\tau_1, \tau_2, \ldots$:

$$g(x/\tau_1, \tau_2, \ldots) = \prod_{j=1}^{\infty} M\,\{z^{n_j}/\tau_1, \tau_2, \ldots\} =$$
$$= \prod_{j=1}^{\infty} \{zP_j(\tau^{(j)}) + [1 - P_j(\tau^{(j)})]\}. \tag{7}$$

By substituting Eq. (5) into Eq. (7), we obtain

$$g(x/\tau_1, \tau_2, \ldots) =$$
$$= \prod_{j=1}^{\infty} \left\{zt\ \left(-\frac{\tau^{(j)}}{\sigma}\right) + \left[1 - t\ \left(-\frac{\tau^{(j)}}{\sigma}\right)\right]\right\}. \tag{8}$$

To obtain Eq. (6), it is necessary to average Eq. (8) over $\tau_1, \tau_2, \ldots$. Since $\tau_1, \tau_2, \ldots$ are independent (in the interval $(k\Delta t, (k + 1)\Delta t)$, the occurrence of events at the time moment t_i^0 is a Poisson flux with the fixed intensity $\lambda^{(k)} = \lambda_1$ or $\lambda^{(k)} = \lambda_2$),

$$p(\tau_1, \tau_2, \ldots) = \prod_{j=1}^{\infty} p(\tau_j) = \prod_{j=1}^{\infty} \lambda^{(k)} \exp\{-\lambda^{(k)}\,\tau_j\}. \tag{9}$$

First of all, let us average Eq. (8) over $\tau_2, \tau_3, \ldots$. As a result we obtain

$$g(x/\tau^{(1)}) = \psi(\tau^{(1)}) =$$

$$= \left\{zt\left(-\frac{\tau^{(1)}}{\sigma}\right) + \left[1 - t\left(-\frac{\tau^{(1)}}{\sigma}\right)\right]\right\} \int_0^\infty \int_0^\infty \ldots \times$$

$$\times \prod_{j=2}^{\infty} \left\{zt\left(-\frac{\tau^{(j)}}{\sigma}\right) + \left[1 - t\left(-\frac{\tau^{(j)}}{\sigma}\right)\right]\right\} \times$$

$$\times \prod_{j=2}^{\infty} p(\tau_j)\, \mathrm{d}\tau_2\, \mathrm{d}\tau_3 \ldots =$$

$$= \left\{zt\left(-\frac{\tau^{(1)}}{\sigma}\right) + \left[1 - t\left(-\frac{\tau^{(1)}}{\sigma}\right)\right]\right\} \times$$

$$\times \int_0^\infty p(\tau_2) \left\{\left\{zt\left(-\frac{\tau^{(2)}}{\sigma}\right) + \left[1 - t\left(-\frac{\tau^{(2)}}{\sigma}\right)\right]\right\} \times\right.$$

$$\times \int_0^\infty \int_0^\infty \ldots \prod_{j=3}^{\infty} \left\{zt\left(-\frac{\tau^{(j)}}{\sigma}\right) + \left[1 - t\left(-\frac{\tau^{(j)}}{\sigma}\right)\right]\right\} \times$$

$$\left.\times \prod_{j=3}^{\infty} p(\tau_j)\, \mathrm{d}\tau_3\, \mathrm{d}\tau_4 \ldots \right\} \mathrm{d}\tau_2 =$$

$$= \left\{zt\left(-\frac{\tau^{(1)}}{\sigma}\right) + \left[1 - t\left(-\frac{\tau^{(1)}}{\sigma}\right)\right]\right\} \times$$

$$\times \int_0^\infty p(\tau_2)\, \psi(\tau^{(2)})\, \mathrm{d}\tau_2. \qquad (10)$$

Thus, we find that

$$\psi(\tau^{(1)}) = \left\{zt\left(-\frac{\tau^{(1)}}{\sigma}\right) + \left[1 - t\left(-\frac{\tau^{(1)}}{\sigma}\right)\right]\right\} \times$$

$$\times \int_0^\infty \psi(\tau^{(2)})\, p(\tau_2)\, \mathrm{d}\tau_2. \qquad (11)$$

Let us denote $f(\tau^{(1)}) = \{zt\ (-\tau^{(1)}/\sigma) + [1 - t\ (-\tau^{(1)}/\sigma)]\}$. Then, substituting $p(\tau_2)$ from Eq. (9) into the Eq. (11), we have

$$\psi(\tau^{(1)}) = \varphi(\tau^{(1)}) \int_{\tau^{(1)}}^{\infty} \psi(\tau^{(2)}) \exp\{-\lambda^{(k)}\tau^{(2)}\}\, \mathrm{d}\tau^{(2)}, \qquad (12)$$

where $\varphi(\tau^{(1)}) = f(\tau^{(1)})\, \lambda^{(k)} \exp\{\lambda^{(k)}\tau^{(1)}\}$. By differentiating Eq. (12) with respect to $\tau^{(1)}$, we have

$$\psi'(\tau^{(1)}) = \varphi'(\tau^{(1)}) \int_{\tau^{(1)}}^{\infty} \psi(\tau^{(2)}) \exp\{-\lambda^{(k)}\tau^{(2)}\}\, \mathrm{d}\tau^{(2)} -$$

$$- \psi(\tau^{(1)})\, \varphi(\tau^{(1)}) \exp\{-\lambda^{(k)}\tau^{(1)}\}. \qquad (13)$$

Substituting the expression for the integral from Eq. (12) into the Eq. (13) and making simple calculations, we obtain

$$\mathrm{d}\ln\psi(\tau^{(1)}) = \mathrm{d}\ln\varphi(\tau^{(1)}) -$$

$$- \varphi(\tau^{(1)}) \exp\{-\lambda^{(k)}\tau^{(1)}\}\, \mathrm{d}\tau^{(1)}. \qquad (14)$$

Integration of Eq. (14) over the interval from zero to $\tau^{(1)}$ leads to the following expression for $\psi(\tau^{(1)})$:

$$\psi(\tau^{(1)}) = \frac{\psi(0)}{\varphi(0)}\, \varphi(\tau^{(1)}) \times$$

$$\times \exp\left\{-\int_0^{\tau^{(1)}} \varphi(y) \exp(-\lambda^{(k)} y)\, \mathrm{d}y\right\}. \qquad (15)$$

Note that Eq. (10) implies that

$$\lim_{\tau^{(1)}\to\infty} \psi(\tau^{(1)}) = \lim_{\tau^{(1)}\to\infty} \left\{zt\left(-\frac{\tau^{(1)}}{\sigma}\right) + \left[1 - t\left(-\frac{\tau^{(1)}}{\sigma}\right)\right]\right\} \times$$

$$\times \lim_{\tau^{(1)}\to\infty} \int_0^\infty \left\{zt\left(-\frac{\tau^{(2)}}{\sigma}\right) + \left[1 - t\left(-\frac{\tau^{(2)}}{\sigma}\right)\right]\right\} \times$$

$$\times p(\tau_2)\, \mathrm{d}\tau_2 \ldots = 1 \int_0^\infty p(\tau_2)\, \mathrm{d}\tau_2 \int_0^\infty p(\tau_3)\, \mathrm{d}\tau_3 \ldots = 1. \qquad (16)$$

Let us make the designation $q = \psi(0)/\varphi(0)$. Then, substituting the expression for $\varphi(\cdot)$ from Eq. (12) into the Eq. (15), we have

$$\psi(\tau^{(1)}) = C\, \lambda^{(k)} f(\tau^{(1)}) \times$$

$$\times \exp\left\{-\lambda^{(k)}\left[\int_0^{\tau^{(1)}} [f(y) - 1]\, \mathrm{d}y\right]\right\}. \qquad (17)$$

The constant C can be determined from the boundary condition (16):

$$\lim_{\tau^{(1)}\to\infty} \psi(\tau^{(1)}) = C\, \lambda^{(k)} \lim_{\tau^{(1)}\to\infty} f(\tau^{(1)}) \times$$

$$\times \lim_{\tau^{(1)}\to\infty} \exp\left\{-\lambda^{(k)}\left[\int_0^{\tau^{(1)}} [f(y) - 1]\, \mathrm{d}y\right]\right\} = 1.$$

Taking into account that $\lim_{\tau^{(1)}\to\infty} f(\tau^{(1)}) = 1$, we obtain

$$C\, \lambda^{(k)} = \exp\left\{\lambda^{(k)} \int_0^\infty [f(y) - 1]\, \mathrm{d}y\right\}. \qquad (18)$$

Substituting Eq. (18) into Eq. (17), we find

$$\psi(\tau^{(1)}) = f(\tau^{(1)}) \exp\left\{\lambda^{(k)} \int_{\tau^{(1)}}^{\infty} [f(y) - 1]\, \mathrm{d}y\right\}. \qquad (19)$$

From Eq. (10), it follows that, at $\tau^{(1)} = 0$,

$$\psi(0) = \frac{z + 1}{2} \int_0^\infty p(\tau_2) \times$$

$$\times \left\{ \left\{ zt\ \left(-\frac{\tau_2}{\sigma} \right) + \left[1 - t\ \left(-\frac{\tau_2}{\sigma} \right) \right] \right\} \int\limits_0^\infty p(\tau_3) \times \right.$$

$$\left. \times \left\{ zt\ \left(-\frac{\tau_2+\tau_3}{\sigma} \right) + \left[1 - t\ \left(-\frac{\tau_2+\tau_3}{\sigma} \right) \right] \right\} \mathrm{d}\tau_3 \ldots \right\} \mathrm{d}\tau_2 =$$

$$= \frac{z+1}{2} \int\limits_0^\infty \psi(\tau_2)\ p(\tau_2)\ \mathrm{d}\tau_2 =$$

$$= \frac{z+1}{2} \int\limits_0^\infty \psi(y)\ p(y)\ \mathrm{d}y. \qquad (20)$$

The characteristic function (6) itself can be written, when allowing for Eq. (10), in the following form:

$$g(x) = \int\limits_0^\infty g(x/\tau^{(1)})\ p(\tau^{(1)})\ \mathrm{d}\tau^{(1)} =$$

$$= \int\limits_0^\infty \psi(\tau^{(1)})\, p(\tau^{(1)})\ \mathrm{d}\tau^{(1)} = \int\limits_0^\infty \psi(y)\, p(y)\ \mathrm{d}y. \qquad (21)$$

From a comparison of Eqs.(20) and (21), we see that $\psi(0) = \frac{z+1}{2}\, g(x)$, however, at $\tau^{(1)} = 0$, we have $f(0) = \frac{z+1}{2}$, so $\psi(0) = f(0)g(x)$. As a result we obtain that

$$g(x) = \psi(0)/f(0). \qquad (22)$$

On the other hand, it follows from Eq. (19), that at $\tau^{(1)} = 0$:

$$\frac{\psi(0)}{f(0)} = \exp \left\{ \lambda^{(k)} \int\limits_0^\infty [f(y) - 1]\ \mathrm{d}y \right\}. \qquad (23)$$

Thus, combining Eqs. (22) and (23), we obtain the following expression for the characteristic function:

$$g(x) = \exp \left\{ \lambda^{(k)} \int\limits_0^\infty [f(y) - 1]\ \mathrm{d}y \right\}. \qquad (24)$$

Substituting the expression for $f(\tau^{(1)})$ into Eq. (24) (replacing $\tau^{(1)}$ in it by the integration variable y) and taking into account that $z = \exp(ix)$ and $\lambda^{(k)} \int\limits_0^\infty t\ \left(-\frac{y}{\sigma}\right) \mathrm{d}y = \frac{\lambda^{(k)}\ \sigma}{\sqrt{2\pi}}$, we finally obtain the expression for the characteristic function in the form

$$g(x) = \exp \left\{ \frac{\lambda^{(k)}\ \sigma}{\sqrt{2\pi}}\ (\exp\ (ix) - 1) \right\}. \qquad (25)$$

Let us represent Eq. (25) as an infinite series:

$$g(x) = \exp \left\{ -\frac{\lambda^{(k)}\ \sigma}{\sqrt{2\pi}} \right\} \sum_{n=0}^\infty \frac{[(\lambda^{(k)}\ \sigma/\sqrt{2\pi})\ \exp\ (ix)]^n}{n!} =$$

$$= \sum_{n=0}^\infty \exp\ (ixn)\ \frac{(\lambda^{(k)}\ \sigma/\sqrt{2\pi})^n}{n!} \exp \left\{ -\frac{\lambda^{(k)}\ \sigma}{\sqrt{2\pi}} \right\}. \qquad (26)$$

On the other hand, it follows from Eq. (6) that

$$g(x) = M\ \{\exp\ (ixn)\} = \sum_{n=0}^\infty \exp\ (ixn)\ p(n). \qquad (27)$$

Comparing Eqs. (26) and (27), we obtain

$$p(n) = \frac{(\lambda^{(k)}\ \sigma/\sqrt{2\pi})^n}{n!} \exp \left\{ -\frac{\lambda^{(k)}\ \sigma}{\sqrt{2\pi}} \right\}. \qquad (28)$$

Thus, the probability distribution for the number of transitions through the right boundary $(k + 1)\Delta t$ of the interval $(k\Delta t,\ (k + 1)\Delta t)$ obeys the Poisson law (in other words, the flux of the event transitions through the right boundary of the interval $(k\Delta t,\ (k + 1)\Delta t)$ is also a Poisson flux). Similar proofs can be presented for the left boundary $k\Delta t$ of the interval $(k\Delta t,\ (k + 1)\Delta t)$, as well as for the right boundary $k\Delta t$ of the interval $((k - 1)\Delta t,\ k\Delta t)$ and the left boundary $(k + 1)\Delta t$ of the interval $((k + 1)\Delta t,\ (k + 2)\Delta t)$.

Thus, the final number of events observed in the interval $(k\Delta t,\ (k + 1)\Delta t)$ is

$$r_{k+1} = r^0_{k+1} - r^{0\mathrm{L}}_{k+1} - r^{0\mathrm{R}}_{k+1} + r^{0\mathrm{R}}_{k} + r^{0\mathrm{L}}_{k+2}, \qquad (29)$$

where r^0_{k+1} is the number of events occurring in the interval $(k\Delta t,\ (k + 1)\Delta t)$ at times t^0_i; $r^{0\mathrm{L}}_{k+1}$ is the number of events crossing the left boundary $k\Delta t$ of the interval due to measurement errors and observed in the interval $((k - 1)\Delta t,\ k\Delta t)$; $r^{0\mathrm{R}}_{k+1}$ is the number of events crossing the right boundary $(k + 1)\Delta t$ of the interval $(k\Delta t,\ (k + 1)\Delta t)$ and observed in the interval $((k + 1)\Delta t,\ (k + 2)\Delta t)$; $r^{0\mathrm{R}}_{k}$ is the number of events occurring in the interval $((k - 1)\Delta t,\ k\Delta t)$ at times t^0_i and crossing the right boundary of the interval and observed in the interval $(k\Delta t,\ (k + 1)\Delta t)$; $r^{0\mathrm{L}}_{k+2}$ is the number of events occurring in the interval $((k + 1)\Delta t,$ $(k + 2)\Delta t)$ at times t^0_i and crossing the left boundary of the interval and observed in the interval $(k + 1)\Delta t,\ (k + 2)\Delta t)$.

The probabilities for all these values, but r^0_{k+1}, are determined by Eq. (28). One should only substitute the corresponding number of events for n and the value corresponding to one or other interval for $\lambda^{(k)}$. For instance, for $r^{0\mathrm{L}}_{k+2}$, formula (28) takes the form

$$p(r^{0\mathrm{L}}_{k+2}) = \frac{(\lambda^{(k+1)}\ \sigma/\sqrt{2\pi})^{r^{0\mathrm{L}}_{k+2}}}{r^{0\mathrm{L}}_{k+2}!} \exp \left\{ -\frac{\lambda^{(k+1)}\ \sigma}{\sqrt{2\pi}} \right\}. \qquad (30)$$

The probability for value r^0_{k+1} is determined by usual formula for a Poisson flux.

Now let us suppose that the occurrence time of an event is measured without errors. Then the trajectory of the process $\lambda(t)$, which is a realization of a random process, is a determinate function of time until some moment in time, $k\Delta t$. So the MC-flux can be treated as a non-stationary ordinary Poisson flux of events without any after-effect.[5] Since the errors in measuring the occurrence time of an event are independent, the properties of being ordinary and the absence of an after-effect remain valid in the case when the errors are taken into account. Therefore, the MC-flux of events is again a non-stationary Poisson flux. Then the probability of observing r_{k+1} events of the flux in the interval $(k\Delta t, (k+1)\Delta t)$ is

$$p(r_{k+1}/\Lambda_{k+1}) = \frac{(\Lambda_{k+1})^{r_{k+1}}}{r_{k+1}!} \exp(-\Lambda_{k+1}), \tag{31}$$

where $\Lambda_{k+1} = l\ (r_{k+1})$. The number of events r_{k+1} is determined by the relation (29). Then, taking into account Eq. (28) and Eq. (30), as an example, we obtain

$$\begin{aligned}\Lambda_{k+1} &= \Lambda(\lambda^{(k-1)}, \lambda^{(k)}, \lambda^{(k+1)}) = l\ (r_{k+1}) = \\ &= l\ (r^0_{k+1}) - l\ (r^{0L}_{k+1}) - l\ (r^{0R}_{k+1}) + \\ &+ l\ (r^{0R}_k) + l\ (r^{0L}_{k+2}) = \lambda^{(k)}\,\Delta t - \frac{2\lambda^{(k)}\sigma}{\sqrt{2\pi}} + \\ &+ \frac{\lambda^{(k-1)}\sigma}{\sqrt{2\pi}} + \frac{\lambda^{(k+1)}\sigma}{\sqrt{2\pi}}.\end{aligned} \tag{32}$$

Taking into account Eq. (32), the formula (31) is written in the form

$$\begin{aligned}p(r_{k+1}/\Lambda_{k+1}) &= p(r_{k+1}/\Lambda_{k+1}(\lambda^{(k-1)},\lambda^{(k)},\lambda^{(k+1)})) = \\ &= p(r_{k+1}/\lambda^{(k-1)}, \lambda^{(k)}, \lambda^{(k+1)}) = \frac{1}{r_{k+1}!} \times \\ &\times \left(\lambda^{(k)}\,\Delta t - \frac{2\lambda^{(k)}\sigma}{\sqrt{2\pi}} + \frac{\lambda^{(k-1)}\sigma}{\sqrt{2\pi}} + \frac{\lambda^{(k+1)}\sigma}{\sqrt{2\pi}}\right)^{r_{k+1}} \times \\ &\times \exp\left\{-(\lambda^{(k)}\,\Delta t - \frac{2\lambda^{(k)}\sigma}{\sqrt{2\pi}} + \frac{\lambda^{(k-1)}\sigma}{\sqrt{2\pi}} + \frac{\lambda^{(k+1)}\sigma}{\sqrt{2\pi}})\right\},\end{aligned} \tag{33}$$

which is nothing but the probability entering in Eq. (4) as the second factor. Thus, the transition probability (4) is determined completely.

Considering further the problem stated (i.e., performing calculation procedures) requires imposing certain restrictions on the values Δt and σ. These restrictions follow from formula (33). Considering variants of the formula (they are determined by the values $\lambda^{(k-1)}, \lambda^{(k)}, \lambda^{(k+1)}$; all in all there are eight variants), we come to the following restriction on Δt and σ: $\Delta t > [2(\lambda_1 - \lambda_2)\sigma]/(\lambda_1\sqrt{2\pi})$. On the other hand, the probability of event transitions from the interval $(k\Delta t, (k+1)\Delta t)$ to non-adjacent intervals and, inversely, probability of the event transitions into the interval $(k\Delta t, (k+1)\Delta t)$ from intervals non-adjacent to it must be sufficiently small, so we may require that $\Delta t \le 3\sigma$. Then the restrictions on Δt take the form $[2(\lambda_1 - \lambda_2)\sigma]/(\lambda_1\sqrt{2\pi}) < \Delta t \le 3\sigma$. A concrete choice of the value Δt at the preset values $\lambda_1, \lambda_2, \alpha_1, \alpha_2, \sigma$ may only be performed experimentally, e.g., by way of imitation simulation.

4. DERIVATION OF A RECURSION RELATION FOR THE *A POSTERIORI* PROBABILITY

Let $\mathbf{r}_m = (r_0, r_1, \ldots, r_m)$ be a sequence of events observed during the time interval from 0 to $m\Delta t$ in the subintervals $((k-1)\Delta t, k\Delta t)$ of duration Δt, $k = \overline{0,m}$ ($r_0 = 0$, since the number of events observed in the interval $(-\Delta t, 0)$ at $k = 0$); $\lambda^{(m)} = (\lambda^{(0)}, \lambda^{(1)}, \ldots, \lambda^{(m)})$ is the sequence of unknown (non-observed) values of the process $\lambda(k\Delta t)$ at times $k\Delta t$, $k = \overline{0,m}$ ($\lambda^{(0)} = \lambda(0) = \lambda_1$ or $\lambda^{(0)} = \lambda_2$).

Let us denote the joint probability of the values $\lambda^{(m)}$ and $\mathbf{r}_m$ as $W(\lambda^{(m)}, \mathbf{r}_m)$. Since the flux of the events observed is a Poisson non-stationary flux $W(\lambda^{(m)}, \mathbf{r}_m)$ can be represented as a product of the transition probabilities (4)

$$\begin{aligned}W(\lambda^{(m)}, \mathbf{r}_m) &= W(\lambda^{(0)}, r_0) \prod_{k=1}^{m} p(\lambda^{(k)}/\lambda^{(k-1)}) \times \\ &\times p(r_k/\lambda^{(k-2)}, \lambda^{(k-1)}, \lambda^{(k)}).\end{aligned} \tag{34}$$

Similarly, for $m + 1$

$$\begin{aligned}W(\lambda^{(m+1)}, \mathbf{r}_{m+1}) &= W(\lambda^{(0)}, r_0) \times \\ &\times \prod_{k=1}^{m+1} p(\lambda^{(k)}/\lambda^{(k-1)})\, p(r_k/\lambda^{(k-2)}, \lambda^{(k-1)}, \lambda^{(k)}).\end{aligned} \tag{35}$$

Comparison of Eqs. (34) and (35) leads to the following relation:

$$\begin{aligned}W(\lambda^{(m+1)}, \mathbf{r}_{m+1}) &= W(\lambda^{(m)}, \mathbf{r}_m) \times \\ &\times p(\lambda^{(m+1)}/\lambda^{(m)})\, p(r_{m+1}/\lambda^{(m-1)}, \lambda^{(m)}, \lambda^{(m+1)}).\end{aligned}$$

Hence, taking into account that $W(\lambda^{(m)}/\mathbf{r}_m) = W(\lambda^{(m)}, \mathbf{r}_m)/W(\mathbf{r}_m)$, we obtain

$$\begin{aligned}W(\lambda^{(m+1)}/\mathbf{r}_{m+1}) &= \frac{W(\mathbf{r}_m)}{W(\mathbf{r}_{m+1})} W(\lambda^{(m)}/\mathbf{r}_m) \times \\ &\times p(\lambda^{(m+1)}/\lambda^{(m)})\, p(r_{m+1}/\lambda^{(m-1)},\lambda^{(m)},\lambda^{(m+1)}).\end{aligned} \tag{36}$$

Then the unknown *a posteriori* probability that the state of the process $\lambda(t)$ at time $t = (m+1)\Delta t$ is

$\lambda^{(m+1)}$ ($\lambda^{(m+1)} = \lambda_1$ or $\lambda^{(m+1)} = \lambda_2$) can be written in the form

$$W(\lambda^{(m+1)}/\mathbf{r}_{m+1}) = \sum_{\lambda^{(0)}=\lambda_1}^{\lambda_2} \ldots \sum_{\lambda^{(m)}=\lambda_1}^{\lambda_2} W(\lambda^{(m+1)}/\mathbf{r}_{m+1}). \quad (37)$$

Substituting Eq. (36) into Eq. (37), and making simple transformations we obtain

$$W(\lambda^{(m+1)}/\mathbf{r}_{m+1}) = \frac{W(\mathbf{r}_m)}{W(\mathbf{r}_{m+1})} \times$$
$$\times \sum_{\lambda^{(m-1)}=\lambda_1}^{\lambda_2} \sum_{\lambda^{(m)}=\lambda_1}^{\lambda_2} W(\lambda^{(m-1)},\lambda^{(m)}/\mathbf{r}_m) \times$$
$$\times p(\lambda^{(m+1)}/\lambda^{(m)})\, p(r_{m+1}/\lambda^{(m-1)},\lambda^{(m)},\lambda^{(m+1)}). \quad (38)$$

The probability $W(\lambda^{(m-1)}, \lambda^{(m)}/\mathbf{r}_m)$ can be represented as a product of the probabilities

$$W(\lambda^{(m-1)}, \lambda^{(m)}/\mathbf{r}_m) = W(\lambda^{(m)}/\mathbf{r}_m)\; p(\lambda^{(m-1)}/\lambda^{(m)}, \mathbf{r}_m).$$

Let us consider the probability $p(\lambda^{(m-1)}/\lambda^{(m)},\mathbf{r}_m)$. This is the conditional probability that the process $\lambda(t)$ came to the state $\lambda^{(m)}$ from the state $\lambda^{(m-1)}$ at time $m\Delta t$ provided that of the events have been observed in the interval $(0, m\Delta t)$ ("given the present, find the pastB). As it was already mentioned above, the process $\lambda(t)$ does not depend on the events observed, so $p(\lambda^{(m-1)}/\lambda^{(m)}, \mathbf{r}_m) = p(\lambda^{(m-1)}/\lambda^{(m)})$, i.e., this is the transition probability for the process $\lambda(t)$ directed back into the past. To determine the probability $p(\lambda^{(m-1)}/\lambda^{(m)})$, we use the Bayes formula.[4] For definiteness, let us assume that $\lambda^{(m-1)} = \lambda_1$, $\lambda^{(m)} = \lambda_2$. Then we have

$$p(\lambda^{(m-1)} = \lambda_1/\lambda^{(m)} = \lambda_1) =$$
$$= p(\lambda^{(m)} = \lambda_1/\lambda^{(m-1)} = \lambda_1)\ \pi_1/[p(\lambda^{(m)} =$$
$$= \lambda_1/\lambda^{(m-1)} = \lambda_1)\ \pi_1 + p(\lambda^{(m)} = \lambda_1/\lambda^{(m-1)} = \lambda_2)\ \pi_2], \quad (39)$$

where π_1, π_2 are probabilities defined above in Sect. 2. Other probabilities are transition probabilities of the process $\lambda(t)$. They are directed to the future from the present and are defined in the formula (4). Substituting these expressions into Eq. (39) and changing the indices, we obtain

$$p(\lambda^{(m-1)} = \lambda_1/\lambda^{(m)} = \lambda_1) = \pi_1 + \pi_2 \exp\{-(\alpha_1 + \alpha_2)\,\Delta t\}. \quad (40)$$

Comparison of the expression (40) with the formula for the transition probability $p(\lambda^{(m)} = \lambda_1/\lambda^{(m-1)} = \lambda_1)$ from Eq. (4) demonstrates their full coincidence, i.e., the process $\lambda(t)$ is reversible. Other transition probabilities are obtained quite similarly:

$$p(\lambda^{(m-1)} = \lambda_2/\lambda^{(m)} = \lambda_1) = \pi_2 - \pi_2 \exp\{-(\alpha_1 + \alpha_2)\,\Delta t\},$$

$$p(\lambda^{(m-1)} = \lambda_1/\lambda^{(m)} = \lambda_2) = \pi_1 - \pi_1 \exp\{-(\alpha_1 + \alpha_2)\,\Delta t\},$$

$$p(\lambda^{(m-1)} = \lambda_2/\lambda^{(m)} = \lambda_2) = \pi_2 + \pi_1 \exp\{-(\alpha_1 + \alpha_2)\,\Delta t\}. \quad (41)$$

Taking into account what we have just said above, one can write Eq. (38) in the form

$$W(\lambda^{(m+1)}/\mathbf{r}_{m+1}) = \frac{W(\mathbf{r}_m)}{W(\mathbf{r}_{m+1})} \times$$
$$\times \sum_{\lambda^{(m-1)}=\lambda_1}^{\lambda_2} \sum_{\lambda^{(m)}=\lambda_1}^{\lambda_2} W(\lambda^{(m)}/\mathbf{r}_m)\; p(\lambda^{(m-1)}/\lambda^{(m)}) \times$$
$$\times p(\lambda^{(m+1)}/\lambda^{(m)})\, p(r_{m+1}/\lambda^{(m-1)},\lambda^{(m)},\lambda^{(m+1)}). \quad (42)$$

Now it only remains to determine the unknown factor $W(\mathbf{r}_m)/W(\mathbf{r}_{m+1})$ which can be obtained from the normalization condition

$$\sum_{\lambda^{(m+1)}=\lambda_1}^{\lambda_2} W(\lambda^{(m+1)}/\mathbf{r}_{m+1}) = \frac{W(\mathbf{r}_m)}{W(\mathbf{r}_{m+1})} \times$$
$$\times \sum_{\lambda^{(m-1)}=\lambda_1}^{\lambda_2} \sum_{\lambda^{(m)}=\lambda_1}^{\lambda_2} \sum_{\lambda^{(m+1)}=\lambda_1}^{\lambda_2} W(\lambda^{(m)}/\mathbf{r}_m) \times$$
$$\times p(\lambda^{(m-1)}/\lambda^{(m)})\; p(\lambda^{(m+1)}/\lambda^{(m)}) \times$$
$$\times p(r_{m+1}/\lambda^{(m-1)},\lambda^{(m)},\lambda^{(m+1)}) = 1.$$

Thus,

$$\frac{W(\mathbf{r}_m)}{W(\mathbf{r}_{m+1})} = \left\{ \sum_{\lambda^{(m-1)}=\lambda_1}^{\lambda_2} \sum_{\lambda^{(m)}=\lambda_1}^{\lambda_2} \sum_{\lambda^{(m+1)}=\lambda_1}^{\lambda_2} W(\lambda^{(m)}/\mathbf{r}_m) \times \right.$$
$$\times p(\lambda^{(m-1)}/\lambda^{(m)})\; p(\lambda^{(m+1)}/\lambda^{(m)}) \times$$
$$\left. \times p(r_{m+1}/\lambda^{(m-1)}, \lambda^{(m)}, \lambda^{(m+1)}) \right\}^{-1}. \quad (43)$$

Substituting Eq. (43) into Eq. (42), we finally obtain the recursion formula for the *a posteriori* probability $W(\lambda^{(m+1)}/\mathbf{r}_{m+1})$:

$$W(\lambda^{(m+1)}/\mathbf{r}_{m+1}) = \left\{ \sum_{\lambda^{(m-1)}=\lambda_1}^{\lambda_2} \sum_{\lambda^{(m)}=\lambda_1}^{\lambda_2} W(\lambda^{(m)}/\mathbf{r}_{m+1}) \times \right.$$
$$\times p(\lambda^{(m-1)}/\lambda^{(m)})\; p(\lambda^{(m+1)}/\lambda^{(m)}) \times$$
$$\left. \times p(r_{m+1}/\lambda^{(m-1)}, \lambda^{(m)}, \lambda^{(m+1)}) \right\} \times$$

$$\times \left\{ \sum_{\lambda^{(m-1)}=\lambda_1}^{\lambda_2} \sum_{\lambda^{(m)}=\lambda_1}^{\lambda_2} \sum_{\lambda^{(m+1)}=\lambda_1}^{\lambda_2} W(\lambda^{(m)}/\mathbf{r}_m) \times \right.$$

$$\times\, p(\lambda^{(m-1)}/\lambda^{(m)})\; p(\lambda^{(m+1)}/\lambda^{(m)}) \times$$

$$\left. \times\, p(r_{m+1}/\lambda^{(m-1)}, \lambda^{(m)}, \lambda^{(m+1)}) \right\}^{-1}. \quad (44)$$

The formula (44) enables one to calculate the *a posteriori* probability at any time $m\Delta t$, $m = 0, 1, \ldots$. Here, for $m = 0$, $W(\lambda^{(0)} = \lambda_1/r_0) = \pi_1$, $W(\lambda^{(0)} = \lambda_2/r_0) = \pi_2$, $\lambda^{(-1)} = \lambda_1$ or $\lambda^{(-1)} = \lambda_2$.

In conclusion, we should like to note that the approach usually used in problems of optimal non-linear filtration[2,3] assumes a limiting transition, in Eq. (44), at $\Delta t \to 0$. The transition leads to a differential equation for the *a posteriori* probability. However, in our case when the measurement errors are taken into account, such a transition to limit is incorrect because the probabilities (33) can be negative as $\Delta t \to 0$. Making the limiting transition in Eq. (44) is only possible if $\sigma \to 0$ simultaneously with Δt. This means that the case when the occurrence time of an event is measured without errors is the limiting case. It was considered in Ref. 3.

5. RESULTS OF NUMERICAL CALCULATIONS

To obtain numerical results, we have developed an algorithm for calculating the *a posteriori* probability by Eq. (44). The calculation program is written in Pascal algorithmic language. The first stage of calculations includes imitation simulation of the MC-flux of events and simulation of the mechanism of errors origin in measurements of occurrence time of an event. We omit the description of the imitation simulation algorithm because it does not contain any difficulties. The second stage is a straightforward calculation of the *a posteriori* probability by formula (44).

All the calculations are being performed for the following values of the parameters: $\lambda_1 = 10$, $\lambda_2 = 1$, $\alpha_1 = 0.5$, $\alpha_2 = 0.7$, $\sigma = 0.05$. Figures 2 to 4 present values of the *a posteriori* probability $W(\lambda_1/t)$ for $\Delta t = 0.0359$ (see Fig. 2); 0.09295 (Fig. 3); 0.15 (Fig. 4).

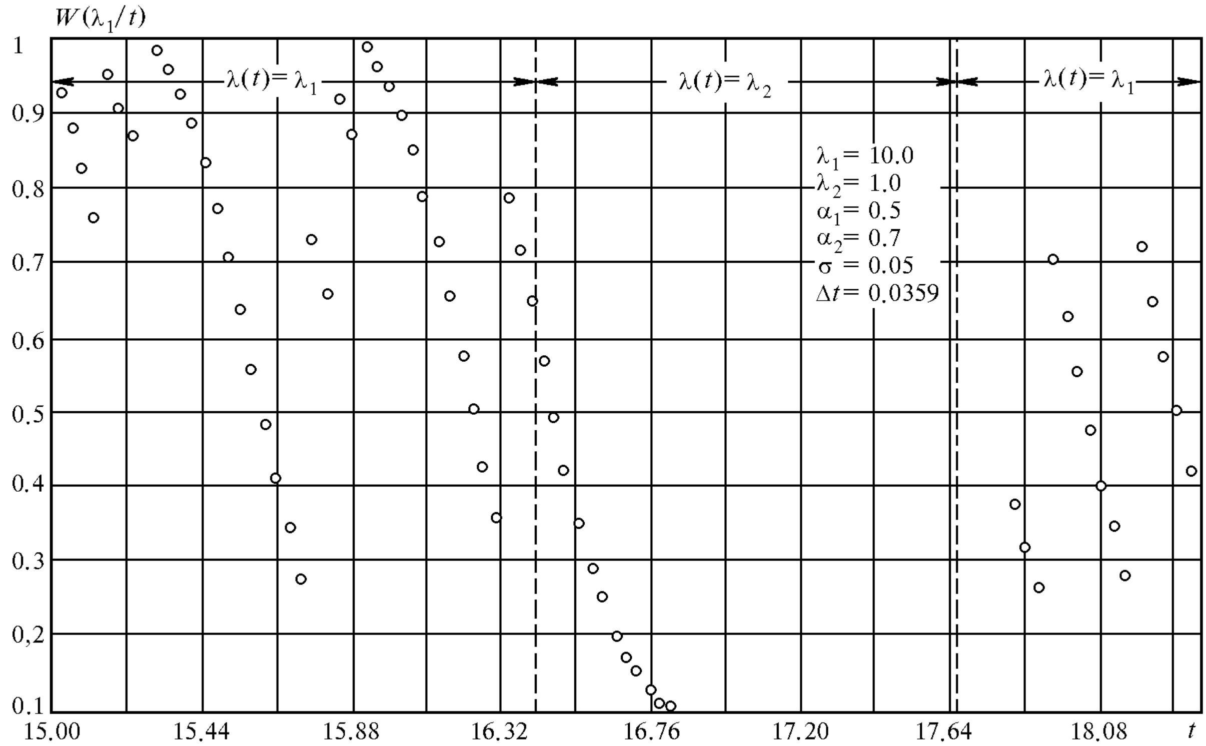

FIG. 2.

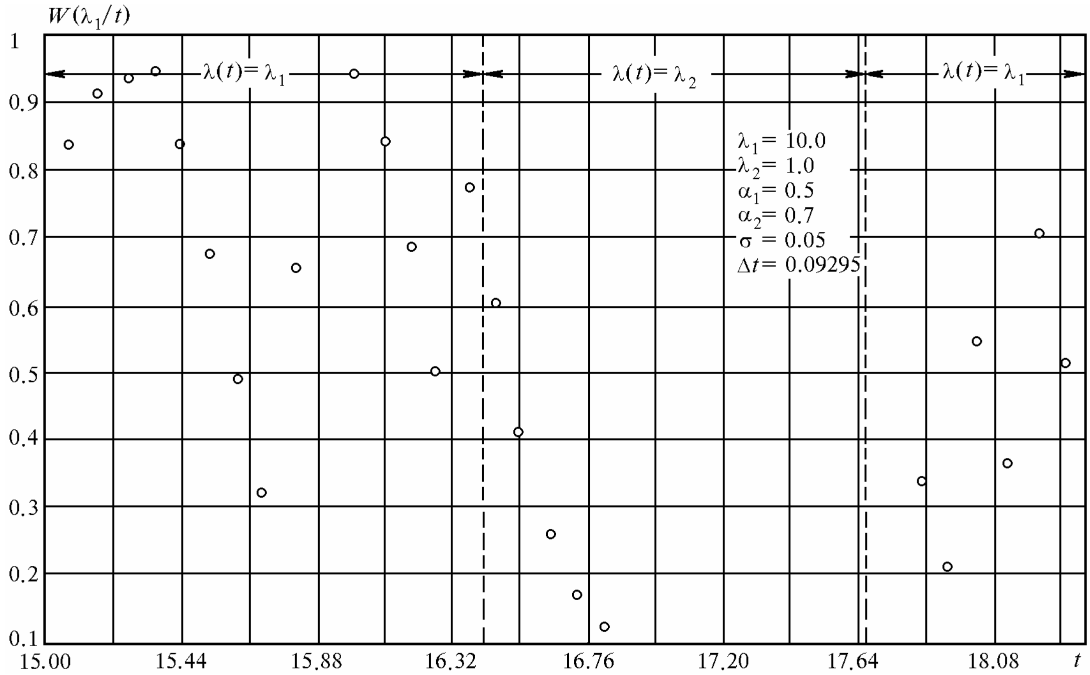

FIG. 3.

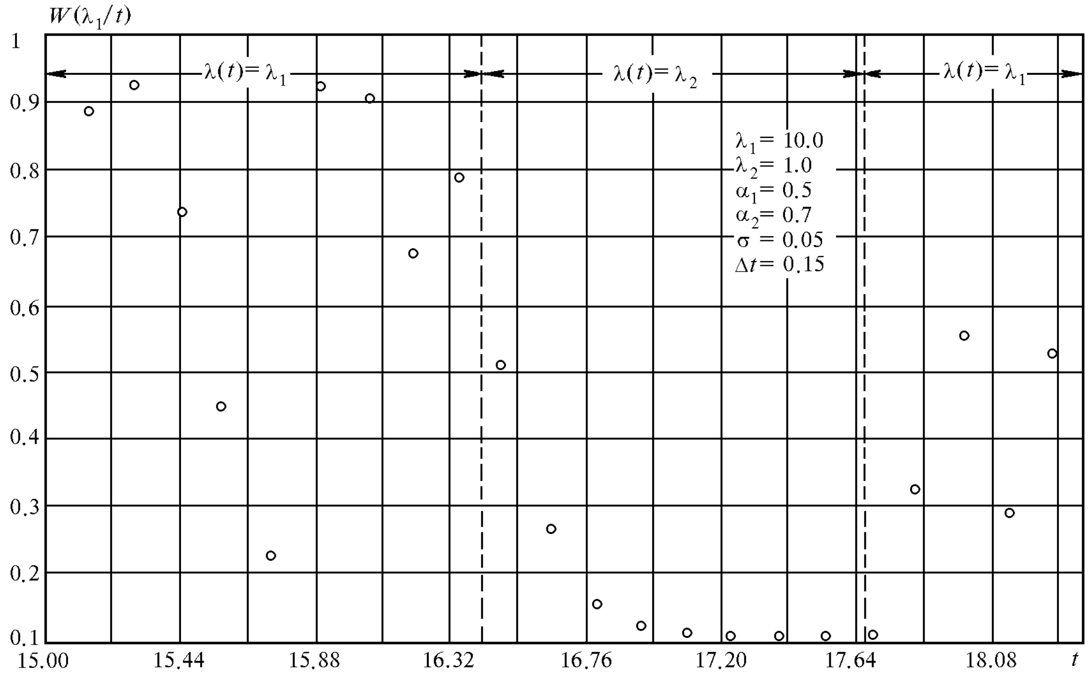

FIG. 4.

The figures demonstrate the intervals for which the process $\lambda(t)$ is stationary. To determine a tolerable value of Δt for given parameters, a statistical experiment has been performed. For each Δt, 50 realizations of the MC-flux with measurement errors at the moments of the events' occurrence were imitated. Duration of each realization was $t = 50$ units. At time t, the probability $W(\lambda_1/t)$ has been calculated, and the decision about one or other state of the process $\lambda(t)$ is made using the criterion of maximum of the *a posteriori* probability. Then, the solution obtained was compared with the true state the process $\lambda(t)$ had at time t known from the results of imitation simulation, and frequency of wrong decisions has also been calculated:

$$P_1 = \hat{P}(\lambda(t) = \lambda_1/\lambda(t) = \lambda_2) = n_1/N_2,$$

$$P_2 = \hat{P}(\lambda(t) = \lambda_2/\lambda(t) = \lambda_1) = n_2/N_1,$$

where N_i is the number of realizations, for which the true state of the process $\lambda(t)$ at time t is λ_i, $i = 1, 2$, $N = N_1 + N_2$; n_1 is the number of wrong decisions about the second state of the process $\lambda(t)$ (at time t, the state of the process is $\lambda(t) = \lambda_2$ but the wrong decision $\lambda(t) = \lambda_1$ is made); n_2 is the number of wrong decisions about the first state of the process $\lambda(t)$ (at time t, the state of the process is $\lambda(t) = \lambda_1$ but the wrong decision $\lambda(t) = \lambda_2$ is made). The estimate of the total probability of an error was calculated by the formula $p = \pi_1 p_2 + \pi_2 p_1$. The results of the statistical experiment are presented in the Table I.

TABLE I.

P_i	$\Delta t = 0.0359$	$\Delta t = 0.09295$	$\Delta t = 0.15$
P_1	0.077	0.077	0.038
P_2	0.250	0.292	0.333
P	0.178	0.202	0.210

Analysis of the results demonstrates that, for the parameters given, the total probability of an error is smaller the closer the value Δt is to the left boundary $2(\lambda_1 + \lambda_2)\sigma/\lambda_1\sqrt{2\pi} = 0.035905$. However, to obtain a tolerable value of Δt, when varying the initial parameters, it is necessary to perform the whole statistical experiment again.

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