ON RECONSTRUCTION OF THE EXTINCTION COEFFICIENT PROFILE OF A MEDIUM FROM ITS OPTICAL TRANSFER FUNCTION TAKING INTO ACCOUNT MULTIPLE SCATTERING WITHIN THE SMALL–ANGLE APPROXIMATION

V.V. Veretennikov

Institute of Atmospheric Optics,
Siberian Branch of the Russian Academy of Sciences, Tomsk
Received December 30, 1992

A technique is proposed for reconstructing spatial distribution of the extinction coefficient of an inhomogeneous coursely disperse medium from its optical transfer function. The technique is based on solving the radiation transfer equation in the small angle approach. Approximate expressions are derived for estimating the distance to a scattering layer and its thickness. A regularization algorithm for solving the inverse problems is described and its precision characteristics are studied in a numerical experiment.

INTRODUCTION

The optical methods are effective for studying a structure of disperse media including an aerosol component of the atmosphere. Of the available methods for optical diagnostics of aerosol media, most of them are based on the phenomenon of single light scattering and due to this fact they are confined to the region of a weakly turbid atmosphere. With increase of the optical density of a disperse medium the effects of multiple light scattering come into play. An account of them in diagnostics is a problem of great concern.

The methods of lidar sounding[1,2] as well as the methods of transmission tomography[3,4] of the atmosphere can be used for studying a spatial structure of the atmospheric aerosol component. In the lidar methods the effect of multiple scattering in the measured lidar returns can be taken into account using its numerical estimate based on a solution of the radiation transfer equation (RTE) for different geometries of the experiment and optical models of the atmosphere with subsequent corrections of the lidar equation.[5,6] In so doing only a portion of the lidar return caused by single scattering, whose contribution decreases with increase of turbidity of a medium, is actually used for the interpretation. The difference between the total and singly scattered signals is represented as the disturbance to be considered.

In the method of diagnostics of optically dense disperse media[7] the multiply scattered radiation is treated not as noise but as an informative component of the measured return signal used for the interpretation. The method[7] is based on the RTE solution in small–angle approximation which enables one to determine the analytical relationship between a disperse composition of the medium and angular distribution of a multiply scattered plane wave.

In this paper a small–angle approximation of the RTE is analyzed as applied to the problem of reconstructing a spatial structure of the strongly turbid aerosol atmosphere from the data on its optical transfer function (OTF). The OTF of the medium is a Fourier image of the other important feature of the medium, i.e., the point spread function (PSF), and is of first importance in the image transfer in scattering media. The problems of theoretical and experimental determination of the OTF (PSF) have been considered in detail, e.g., in Refs. 8 and 9. The OTF of a medium depends on both the scattering phase function characterizing local properties of the scattering volume and spatial distribution of scattering and extinction coefficients over the direction of radiation propagation. The study of possibilities for reconstruction of the extinction coefficient profile of a medium from the data on its OTF has received primary attention in this paper. It should also be noted that the information about the scattering phase function described by the OTF makes it possible to consider the other inverse problem in reconstructing the disperse composition of the medium. In contrast to the method[7] based on measurements of an angular structure of multiply scattered radiation, in the latter case the initial information for solving the inverse problem can be extracted by measuring the spatial distribution of illumination in the cross section of a narrow light beam propagated through the scattering medium. This problem is of particular importance and will be treated in the other paper.

1. Initial equations. Mathematical statement of the inverse problem. The OTF of the medium $F(\nu, z)$ can be derived from the general solution of the RTE in small–angle approximation within a frequency range in the form[10]

$$F(\nu, z) = \exp\{-\tau(z) + g(\nu)\}, \tag{1}$$

where

$$\tau(z) = \int_0^z \varepsilon(s)\, ds, \tag{2}$$

$$g(\nu) = \int_0^z \sigma(z - s)\, x(\nu s)\, ds, \tag{3}$$

where ν is the spatial frequency, σ and ε are the scattering and extinction coefficients, τ is the optical depth of the medium in the interval $[0, z]$, and $x(p)$ is the Fourier transform of a small–angle scattering phase function. The function $x(p)$ in the Fraunhofer diffraction approximation, with an accuracy of the scale, coincides with a medium–sized autocorrelation function of a particle shadow $\varphi(\rho)$ related to the area of the cross section of particles:

0235-6880/93/04 250-05 $02.00

$x(p) = \varphi(p/\kappa)$ and $\kappa = 2\pi/\lambda$, where λ is the wavelength. For a polydisperse ensemble of spherical scatterers the autocorrelation function $\varphi(\rho)$ can be expressed as an integral[7]

$$\varphi(\rho) = \int_{\rho/2}^{R} Q\,(\rho,\ r)\ f(r)\ \mathrm{d}r, \qquad (4)$$

where $f(r) = s(r)/S$, $s(r) = \pi r^2 n(r)$, $n(r)$ is the particle size distribution, $S = \int_0^R s(r)\ \mathrm{d}r$ is the total geometric cross section of particles in the unit volume of the scattering medium; $Q(\rho,\ r) = G(\rho/2r)$ is the autocorrelation function of a shadow of a single r–radius particle related to the area of its cross section

$$G(t) = \begin{cases} 2\pi^{-1}\left[\arccos t - t\sqrt{1-t^2}\ \right], & t \le 1, \\ 0\ , & t > 1\ . \end{cases} \qquad (5)$$

Relations (1)–(4) are valid under conditions[11] $\kappa r|m - 1| \gg 1$ (m is the complex refractive index of particles of a medium) and $\tau < 8$. These limitations are related to the total conditions of applicability of small–angle approximation of the RTE (Refs. 8 and 12) and special representation of Fourier transform of the scattering phase function in terms of Eq. (4).

Under the aforementioned assumptions the relations $\varepsilon = 2S$ and $\sigma = S$ are valid and the function $g(\nu)$ from Eq. (3), and, hence, the OTF of the medium $F(\nu)$ are directly expressed in terms of microstructure parameters of the medium by the integral transformation

$$g(\nu) = \frac{1}{2}\int_0^z \int_0^R G(\nu\, s/2\,\kappa\, r)\ \varepsilon(z - s)\ f(r)\ \mathrm{d}r\ \mathrm{d}s\ . \qquad (6)$$

Relation (6) can be treated as an integral equation for determining the normalized particle–size spectrum $f(r)$ (with the known profile $\varepsilon(s)$) or for reconstructing the extinction coefficient profile $\varepsilon(s)$ with an *a priori* specified form of the particle size distribution. In what follows we will consider the latter problem as represented in a standard form of the first–type integral equation

$$\int_0^z K(\nu,\ s)\ \tilde{\varepsilon}(s)\ \mathrm{d}s = g(\nu) \qquad (7)$$

with respect to the function $\tilde{\varepsilon}(s) = \varepsilon(z - s)$ with the kernel $K(\nu,\ s) = \varphi(\nu\, s/\kappa)/2$. The initial information in the case of inversion of integral equation (7) is prescribed by the function $g(\nu)$ which is unambiguously expressed in terms of the OTF of a medium

$$g(\nu) = \ln F(\nu) + \tau\ . \qquad (8)$$

Taking into account properties of the function $G(t)$ it is possible to show that $\varphi(0) = 1$ and $F(0) = \exp(-\tau/2)$ and the right side of integral equation (7) is represented in the final form

$$g(\nu) = \ln\left(F(\nu)/F^{\,2}(0)\right)\ . \qquad (9)$$

2. Structure analysis of the inverse problem. Let us consider some analytical properties of the kernel $K(\nu,\ s)$ of integral equation (7) which determine principal peculiarities of the $g(\nu)$ function behavior (and hence the behavior of the OTF of a medium $F(v)$) and specify the information content of the OTF measurements with respect to spatial distribution of the extinction coefficient $\varepsilon(s)$. Reasoning from the properties of the functions $G(t)$ and $f(r)$ it is easy to show that the kernel $K(\nu,\ s)$ is a continuous monotonically descreasing convex downwards function over each of arguments ν and s with the domain $0 \le s \le z$, $0 \le \nu \le \infty$ and the variability region $0 \le K(\cdot) \le 0.5$. Let us determine the value of the frequency $\nu_0 = 2\kappa R/z$. Then with $\nu \ge \nu_0$ the kernel vanishes $K(\nu,\ s) = 0$ for all the values of s from the interval $[s_1,\ z]$, where $s_1 = z\,\nu_0/\nu$. Hence it follows that with $\nu \ge \nu_0$ the upper integration limit in Eq. (7) is equal to s_1, i.e., it is a function of the frequency ν. Therefore for the values of $\nu \ge \nu_0$ the function $g(\nu)$ does not contain any information about distribution of $\tilde{\varepsilon}(s)$ in the interval $[s_1,\ z]$. Since $\tilde{\varepsilon}(s) = \varepsilon(z - s)$, the value of $g(\nu)$ for $\nu \ge \nu_0$ is determined by the values of the extinction coefficient $\varepsilon(s)$ in the interval $[z - s_1,\ z]$, i.e., in the near zone to the receiver. Thus with the frequency increase starting from $\nu = \nu_0$, the region of the scattering volume located at a distance larger than s_1 from the receiver does not affect the OTF formation of a medium.

It follows from the foregoing properties of the kernel $K(\nu,\ \tau)$ and nonnegative value of the extinction coefficient $\varepsilon(\sigma)$ that $g(\nu)$ is a nonnegative, monotonically decreasing convex downwards function for $0 \le \nu \le \infty$

$$0 \le g(\nu) \le \tau/2;\quad g'(\nu) \le 0;\quad g''(\nu) \ge 0.$$

It should be noted that if the scattering layer is at a certain distance H from the receiver, then $g(\nu) = 0$ for $\nu > \nu_{\max} = 2\kappa R/\mathrm{H}$. By this is meant that the spectrum of frequencies, for which the OTF of the scattered component is nonzero, is limited. The spectrum width varies in direct proportion to the scattering particle size and in inverse proportion to the distance to the layer. This allows one, based on the spectrum width, to estimate the distance to the near boundary of the scattering layer. It is possible to obtain the estimate of the threshold frequency $\nu^*_{\max}$ from the condition $g(\nu^*_{\max}) = \sigma$, where σ is a sufficiently small positive value. In this case the corresponding estimate of the distance to the layer H^* is overestimated. For example, for the Gaussian model of the layer with the optical thickness $\tau = 1$, whose near boundary is at a 7.1 km distance from the receiver, the estimate of H^* is 7.8 km for $\sigma = 0.01$.

The important information about the layer structure is extracted from the analysis of the slope angle of the curve $g(\nu)$ at zero. Taking into account the expression for the derivative

$$G'(t) = -\,(2/\pi)\sqrt{1 - t^{\,2}},$$

differentiating Eq. (7) over v, and assuming $\nu = 0$ in the obtained expression, we have

$$g'(0) = -\frac{A}{\kappa\pi}\int_0^z \varepsilon\,(z - s)\ s\ \mathrm{d}s, \qquad (10)$$

where the multiplier A is determined by a microstructure of a medium

$$A = \int_0^R r^{-1} f(r)\, \mathrm{d}r \tag{11}$$

and can be expressed in terms of the integral microstructural parameters in the form

$$A = \pi N \overline{r}_n / S, \tag{12}$$

where N is the particle number density and $\overline{r}(n)$ is the particle radius avereged over the distribution $n(r)$. In the particular case of a monodisperse medium with particles of the radius R we have $A = 1/R$.

By representing the integral in Eq. (10) as

$$T = \int_0^z \varepsilon(s)\,(z - s)\, \mathrm{d}s, \tag{13}$$

it is possible to show that the relation

$$L = T/\tau \tag{14}$$

determines the distance from the receiver to "the center of gravity" of the scattering layer. Thus the layer position along the viewing line can be determined from the value of the derivative $g'(\nu)$ at zero.

Under certain additionally simplifying assumptions formula (10) can serve as a basis for estimating the geometric depth of the scattering layer. Consider the scattering layer of the thickness D whose near boundary is at the distance H from the observer. By determining the efficient value of the extinction coefficient in the layer

$$\varepsilon_0 = \int_H^{H+D} \varepsilon\,(z - s)\, s\, \mathrm{d}s \left[\int_H^{H+D} s\, \mathrm{d}s\right]^{-1}, \tag{15}$$

we have

$$T = \varepsilon_0\, D(H + D/2). \tag{16}$$

By assuming approximately $\tau = \varepsilon_0 \Delta$, from Eq. (16) we can derive the formula for estimating the geometric thickness D of the scattering layer

$$D = 2\,(T/\tau - H), \tag{17}$$

where

$$T = \kappa\pi\, |g'(0)|/A\ .$$

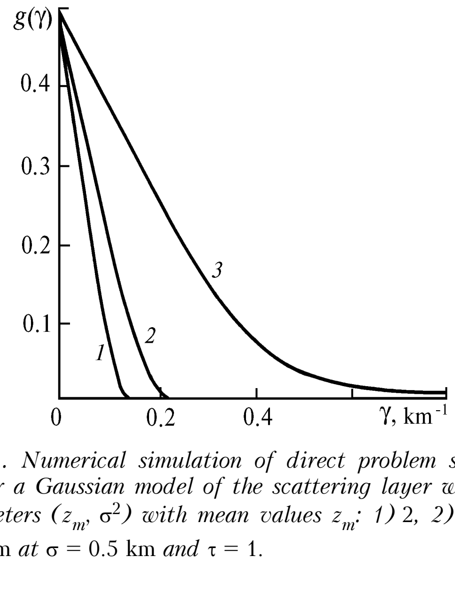

FIG. 1. Numerical simulation of direct problem solution (7) for a Gaussian model of the scattering layer with the parameters (z_m, σ^2) with mean values z_m: 1) 2, 2) 5, and 3) 8 km at $\sigma = 0.5$ km and $\tau = 1$.

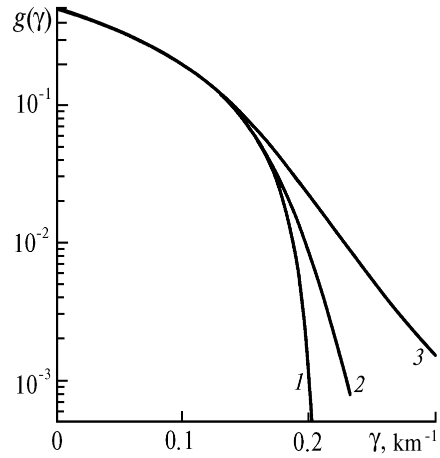

FIG. 2. Numerical simulation of direct problem solution (7) for layers with different geometric thickness (thickness σ: 1) 0.1, 2) 0.5, and 3) 1.0 km at the fixed position of the center ($z_m = 5$ km) and constant optical thickness ($\tau = 1$).

As an example Fig. 1 depicts a plot of calculational results of the function $g(\gamma)$ vs the parameter $\gamma = \mu/2\kappa R$ at three positions of the scattering layer simulated by the normal distribution with the mean values of z_m and variance σ^2. Curves *1–3* correspond to the values of the parameter $z_m = 2.5$ and 8 km at $\sigma = 0.5$ km. As can be seen in Fig. 1 the slope of curves at zero is uniquely determined by the layer position. The effect of the layer thickness characterized by the variance σ^2 at a fixed position of its center ($z_m = 5$ km) is shown in Fig. 2 in which curves *1–3* correspond to the values of the parameter $\sigma = 0.1$, 0.5, and 1.0 km. The results represented in Fig. 2 show that when the layer thickness changes by a factor of 10 the behavior of the function $g(\gamma)$ in the low–frequency range ($\gamma < 0.1$ km^{-1}) remains stable and the characteristic differences appear in the high frequency range ($\gamma > 0.15$ km^{-1}).

To estimate applicability of approximate formulas (14) and (17) the table lists the calculation results of the distance L to "the center of gravity" and the geometric thickness of the scattering layer D when simulating the extinction coefficient profile $\varepsilon(s)$ along the path $0 \le s \le 10$ km using a Gaussian curve with the parameters $z_m = 2.5$ and 8 km, $\sigma = 0.5$ km and $\tau = 1$.

As the table shows the reconstructed geometric thickness of the scattering layer D fits "the 3σ rule" for the normal distribution.

Thus the given examples show that the analysis of the structure of initial data (of the spectrum width and the derivative $g'(\nu)$ at zero) enables one to extract useful information about the properties of the unknown profile of the extinction coefficient $\varepsilon(s)$ (in particular, the distance to the scattering layer, its optical and geometric thickness). The more detailed interpretation of the OTF measurements of a scattering medium is possible when the numerical methods for solving the inverse problems are used.

3. The inversion algorithm and the numerical experiment. The regularizing algorithm developed based on the Tikhonov method[13] was used for inverting integral equation (7). Preliminary algebraization of Eq. (7) was made using the method described in Ref. 14 and based on approximation of the extinction coefficient $\varepsilon(s)$ with a linear spline by the formula

$$\varepsilon(s) = \sum_{j=1}^{n} \varepsilon_j \, N_j\,(s), \tag{18}$$

where the coefficients ε_j determine the values of the unknown function $\varepsilon(s)$ at the nodes $s_j = \Delta(j - 1)$ assigned in the interval $[0, z]$ with an even step $\Delta = z/(n - 1)$. The base functions $N_j(s)$, $j = 1, \ldots, n$ have the form

$$N_j\,(s) = \begin{cases} 1 - \dfrac{|s - s_j|}{\Delta}\,, & |s - s_j| \le \Delta\,, \\ 0\,, & |s - s_j| > \Delta\,. \end{cases} \tag{19}$$

By replacing the function $\varepsilon(s)$ in Eq. (7) by its approximation using formula (18) for a discrete set of frequencies $\nu_i = \nu_{\max}(i - 1)/(m - 1)$, $i = 1, \ldots, m$, we can obtain a system of linear algebraic equations with respect to the vector $\varepsilon = (\varepsilon_n, \ldots, \varepsilon_2, \varepsilon_1)$

$$A\,\varepsilon = g \tag{20}$$

with the matrix $A = \|\,a_{ij}\|$ whose elements are determined by the relations

$$a_{ij} = \int\limits_{s_{j-1}}^{s_{j+1}} K(\nu_i,\ s)\, N_j\,(s)\, \mathrm{d}s,\ i, j = 1, \ldots, n;\ s_0 = 0;\ s_{n+1} = z. \tag{21}$$

The regularized analog of system (20) has the form

$$(A^{\mathrm{T}} E^{-2} A + \alpha\, D)\, \varepsilon = A^{\mathrm{T}} E^{-2} g\ , \tag{22}$$

where A^{T} is the matrix transposed to the A matrix, $E = \mathrm{diag}(e_1, \ldots, e_m)$, e_j are the weighting factors determined by an error in the jth measurement; D is a tridiagonal smoothing matrix with the elements

$$d_{ij} = \frac{1}{\Delta} \begin{cases} 2, & i = j, \\ -1, & |i - j| = 1, \\ 0, & |i - j| > 1,\ \ i, j = 1, \ldots, n. \end{cases} \tag{23}$$

The regularization parameter d was chosen based on the principle of minimum discrepancies[15] which allows one to take into account an *a priori* information about the nonnegative unknown function $\varepsilon(s)$ and the optical thickness (τ) in this problem. According to Ref. 15 the value of α_{md}, with respect to criterion of minimum discrepancies, is determined from the condition of minimum of the functional

$$F_{\mathrm{md}} = \|\, A\ \varepsilon_\alpha - g\ \| + \|\, A\mathrm{e}_\alpha^{(+)} - g\ \|, \tag{24}$$

where $= P_1 P_2 \varepsilon_\alpha$ and P_2 is the operator of projection onto a set of nonnegative functions, and the nonlinear operator P_1 is determined as follows

$$P_1\, \varepsilon_\alpha = \tau\ \varepsilon_\alpha \left[\int\limits_0^z \varepsilon_\alpha(s)\ \mathrm{d}s \right]^{-1}. \tag{25}$$

In numerical experiments the direct problem was solved in the interval $0 \le s \le 10$ km in which the discrete counts of the extinction coefficient profile $\varepsilon_j = \varepsilon(s_j)$ with the step $\Delta = 0.25$ km were assigned. The dimensionality m of the g vector in the right–hand side of Eq. (20) equals 41. We studied the efficiency of reconstruction of the profile $\varepsilon(s)$ simulated by a Gaussian curve with the parameters (z_m, σ^2) as a function of the layer position with respect to the receiver. The accuracy of the inverse problem solution was found to increase when the receiver is approached to the layer. Thus, e.g., if the rms error in reconstruction of the profile $\varepsilon(s)$ was 17% at $z_m = 8$ km, $\sigma = 0.5$ km, and the 5% relative error of input data, then for the layer with $z_m = 3$ km the error in the solution increased up to 30%.

Given in Fig. 3 is the instance of reconstructing a structure of a laminar layered scattering medium whose model is constructed by means of superposition of two normal distributions with the parameters $z_{m_1} = 7.5$, $z_{m_2} = 9.5$ km, and $\sigma = 0.3$ km. In this example the solution vector dimensionality $n = 15$. The relative error in the input data is 10%. The ordinate axis in Fig. 3 is directed from the receiver to the layer and the origin of coordinates coincides with its nearest boundary. The solution of the inverse problem given in Fig. 3 was obtained taking into account a positive sign of ε_α and the known optical thickness τ for an optimal value of the regularization parameter α. The rms error of reconstruction was 10.4%. The use of the minimum discrepancy criterion for choosing the regularization parameter results in the reconstruction error increase within 5%. It should be noted that for the solution obtained directly by inverting Eq. (22) (without taking into account a positive sign and renormalization to the known optical thickness) the error was larger than 20%.

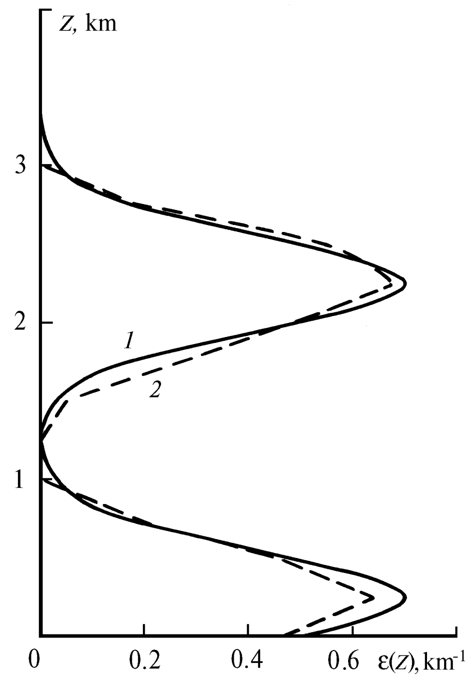

FIG. 3 Example of reconstructing a two–layer model of the extinction profile ε(s) in the numerical experiment: 1) model, 2) solution of the inverse problem with a 10% *error in the initial data and optimal value of the regularization parameter.*

CONCLUSION

Thus in this paper we consider a new approach to the problem in determining the profile of the extinction coefficient of an inhomogeneous scattering medium which uses the information about the optical transfer function of the medium in small–angle approximation of the radiation transfer theory. In contrast to lidar methods the above–discussed problem does not require the pulsed radiation sources to be applied and the resolution of the spatial structure of a medium requires the inverse problem to be solved. For the OTF measurements to be inverted we propose a regularizing algorithm with the use of the *a priori* information about the unknown function. In the numerical experiment we showed the efficiency of the algorithm and studied the efficiency of the inverse problem solution depending on the position of the scattering layer. It was found that to increase the reconstruction accuracy of the extinction coefficient profile it is necessary to carry out measurements at the boundary of the scattering layer. The method can be used for optical diagnostics of a spatial structure of coarsely disperse media of the type of thin cloudy layers or a near–surface layer of the sea under conditions of multiple scattering.

REFERENCES

1. V.E. Zuev, ed., *Remote Methods for Studying the Atmosphere* (Nauka, Novosibirsk, 1980), 160 pp.
2. R.M. Measures, *Laser Remote Sensing* (John Wiley & Sons, New York, 1984).
3. D.C. Wolfe and R.L. Byer, Appl. Opt. **21**, No. 7, 1165–1178 (1982).
4. S. Twomey, J. Atmos. Sci. **44**, No. 23, 3544–3551 (1987).
5. V.A. Korshunov, Atm. Opt. **3**, No. 2, 101–107 (1990).
6. Q. Jinhuan, H. Quenzel, and M. Wigner, in: *Abstracts of Reports at the 15th International Laser Radar Conference,* Institute of Atmospheric Optics, Tomsk (1990), Part 1.
7. N.I. Vagin and V.V. Veretennikov, Izv. Akad. Nauk SSSR, Fiz. Atmos. Okeana **25**, No. 7, 723–731 (1989).
8. E.P. Zege, A.P. Ivanov, and I.L. Katsev, *Image Transfer through the Scattering Medium* (Nauka i Tekhnika, Minsk, 1985), 327 pp.
9. A.N. Valentyuk and K.G. Predko, *An Optical Image during Remote Observation* (Nauka i Tekhnika, Minsk, 1991), 359 pp.
10. D.M. Bravo–Zhivotovskii, L.S. Dolin, A.Y. Luchinin, and V.A. Savel'ev, Izv. Akad. Nauk SSSR, Fiz. Atmos. Okeana **5**, No. 2, 160–167 (1969).
11. V.F. Belov, A.G. Borovoi, N.I. Vagin, and S.N. Volkov, Izv. Akad. Nauk SSSR, Fiz. Atmos. Okeana **20**, No. 3, 323–327 (1984).
12. E.P. Zege and I.K. Katzev, Izv. Akad. Nauk SSSR, Fiz. Atmos. Okeana **27**, No. 2, 1772–181 (1991).
13. A.N. Tikhonov and V.Ya. Arsenin, *Methods for Solving Incorrect Problems* (Nauka, Moscow, 1979), 285 pp.
14. A.G. Borovoi, N.I. Vagin, and V.V. Veretennikov, Opt. Spektrosk. **61**, No. 6, 1326–1330 (1986).
15. V.P. Tanana, *Methods for Solving Operator Equation* (Nauka, Moscow, 1981), 156 pp.