DETERMINATION OF THE ACCURACY OF MEASURING SYSTEMS BY STATISTICAL METHODS

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A method is proposed for determining the error in the estimate of the event flux intensity when a signal is accumulated from realizations in the photon-counting mode. The experimentally obtained dependence of the relative error in the estimate the accumulation level is explained theoretically.

The accuracy of detection of a reflected lidar signal plays an important role in the investigation of atmospheric processes by means of remote sensing. For this reason it is important to know the instrumental error in the estimate of the characteristics of a lidar signal and ways to reduce it. The accuracy of a measuring system (MS) must be determined by the same methods that are used for measuring the lidar signals. This approach makes it possible to reduce substantially the volume of additional apparatus required for these purposes, to automate the entire process, and to eliminate the additional errors, which, in turn, must be measured and monitored in some manner, introduced by this apparatus.[1–3]

When measurements are performed in the photon-counting mode it is necessary to estimate the intensity of the reflected photon flux by accumulating the signal over a large number of realizations. For this reason, a useful method for estimating the measurement error, the analog of the method of photon counting, could be estimation of the intensity of a model Poisson flux, obtained as a result of the accumulation of a signal over N realizations.

Theoretically the detection error can be made as small as desired by increasing the sample volume, since the relative error $\Delta \cong 1/\sqrt{N}$.[4] In practice, however, there is observed a different dependence: as N increases Δ approaches some finite limit and not zero. It is assumed[5] that the existence of this limit is a result of the effect of the electronic channel of the measuring system: fluctuations in the lengths of the time intervals (TIs) in which the events are recorded, instability of the parameters of the photomultiplier, and drift of the switching thresholds of the comparators in the one-electron pulse discriminator, etc. In a rigorous approach it is necessary to develop individual methods for each type of error as well as methods for taking these errors into account and reducing their effect on the accuracy of measuring systems, but this is quite difficult and expensive to do.

In this paper we try to estimate the contribution of the instability of the TIs to the total error in the measurements.

We shall make a number of assumptions that simplify the analysis. It is clear *a priori* that fluctuations of the TIs can be "slow" and "fast." This fact can be expressed by the ratio between the correlation interval τ_c and the observation time T: when $\tau_c \ll T$ the fluctuations are "fast," when $\tau_c \gg T$ the fluctuations are "slow", and, finally, when $\tau_c \sim T$ we observe the intermediate case.

The problem is to determine which fluctuations make a large contribution to the error. We shall determine more accurately the time scales of the correction interval. We assume that the duration of the time intervals is constant for one measurement act and than it fluctuates from one transmission to another. We shall determine the average and the variance of the Poisson flux, when the duration of the time interval T fluctuates.

The duration of the time intervals in the experiments are dependent random quantities. The probability characteristic of the durations of the time intervals is a multidimensional distribution

$$P(\tau) = P(\tau_1, \tau_2 \ldots \tau_N), \quad (1)$$

where τ is a vector with dimension N.

We shall estimate the flux intensity in the observation interval by the usual method used for statistical measurements — by counting the number of events.

$$\hat{\lambda} = \frac{1}{N} \sum_{i=1}^{N} n_i . \quad (2)$$

The average value of the estimate is found in the usual manner:

$$M[\hat{\lambda}] = \frac{1}{N} M[n_i] = \frac{1}{N} \sum_{i=1}^{N} \int \ldots \int P(\bar{T}) d\bar{T} \sum_{n=0}^{\infty} n_i \frac{[\lambda_0 T_i]^{n_i}}{n_i!} \times \exp[-\lambda_0 T_i] .$$

Assuming that the average duration of the time interval T_0 in all readings is identical, we obtain

$$M[\hat{\lambda}] = \lambda_0 T_0 . \quad (3)$$

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where λ_0 is the average intensity of the photocounts.

The second moment of the estimate is

$$M[\hat{\lambda}^2] = \frac{1}{N^2}\left\{N\left(\lambda_0 T_0 + \lambda_0^2\left(\sigma_T^2 + T_0^2\right)\right) + \lambda_0^2 \sum_{\substack{i,j=1\\ i\neq j}}^{N} \int\ldots\int T_i T_j P(T_i, T_j)\,dT_i\,dT_j\right\}$$

We write $\int\ldots\int T_iT_jP[T_i, T_j]dT_idT_j = K_{ij}$. Then

$$M[\hat{\lambda}^2] = \frac{1}{N^2}\left\{N\left(\lambda_0 T_0 + \lambda_0^2\left(\sigma_T^2 + T_0^2\right)\right) + 2\lambda_0^2 \sum_{i<j}^{N} K_{ij}\right\}.$$

The variance of the estimate is

$$D(\hat{\lambda}) = M[\hat{\lambda}^2] - \left(M[\hat{\lambda}]\right)^2 = \frac{1}{N^2}\left\{N\left(\lambda_0 T_0 + \lambda_0^2\left(\sigma_T^2 + T_0^2\right)\right) + 2\lambda_0^2 \sum_{i<j}^{N} K_{ij}\right\} - \lambda_0^2 T_0^2.$$

Using the centered random quantities $\dot{T}_1 = T_1 - T_0$ we obtain

$$K_{ij} = M\left[\left(\dot{T}_i + T_0\right)\left(\dot{T}_j + T_0\right)\right] = M\left[\dot{T}_i\dot{T}_j\right] + T_0^2 = R_{i,j} + T_0^2.$$

$$D(\hat{\lambda}) = \frac{\lambda_0 T_0 + \lambda_0^2\sigma_T^2}{N} + \frac{2\lambda_0^2}{N^2}\sum_{i<j=1}^{N} R_{i,j}. \quad (4)$$

Introducing the normalized correlation coefficient $r_{i,j} = \frac{R_{i,j}}{\sigma_T^2}$ where σ_T^2 is the variance of the duration, we rewrite Eq. (4) as

$$D(\hat{\lambda}) = \frac{\lambda_0 T_0 + \lambda_0^2\sigma_T^2}{N^2} + \frac{2\lambda_0^2\sigma_T^2}{N^2}\sum_{i<j=1}^{N} r_{i,j}. \quad (5)$$

The expression for the relative error in the estimate of the intensity follows from Eqs. (3) and (5)

$$\Delta(\hat{\lambda}) = \frac{1}{\sqrt{\lambda_0 T_0 N}}\sqrt{1 + \frac{\lambda_0\sigma_T^2}{T_0} + \frac{2\lambda_0^2\sigma_T^2}{NT_0}\sum_{i<j=1}^{N} r_{i,j}}. \quad (6)$$

The asymptotic behavior of the relative error depends on the limit to which the parameter $\alpha = \sum_{i,j=1}^{N} r_{i,j}$ approaches.

We shall formulate some conclusions that follow from Eq. (6).

1. In the case of "fast" fluctuations of the duration of the time intervals $r_{i,j} = 0$. Then

$$\Delta(\hat{\lambda}) = \frac{1}{\sqrt{\lambda_0 T_0 N}}\sqrt{1 + \frac{\lambda_0\sigma_T^2}{T_0}},$$

i.e., $\Delta(\hat{\lambda})$ – the relative error in the estimate of the flux intensity of the photo-pulses – can be made as small as desired by increasing the accumulation volume.

2. In the case of "slow" fluctuations it may be assumed that $r_{i,j} = 1$. Then

$$\Delta(\hat{\lambda}) = \frac{1}{\sqrt{\lambda_0 T_0 N}}\sqrt{1 + \frac{N\lambda_0\sigma_T^2}{T_0}}.$$

Here $\Delta(\hat{\lambda})$ has a lower limit determined by the value of σ_T/T_0.

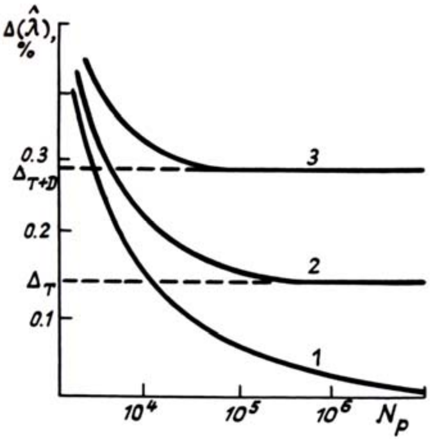

FIG. 1. The dependence of Δ on N_p – the number of photons accumulated: 1) the theoretical dependence $\Delta = f(N_p)$; 2) contribution of fluctuations of the duration of the time intervals Δ_T diagram of the measurements in Fig. 2a; 3) contributions of fluctuations of the duration of the time intervals + fluctuations of the discrimination thresholds and the parameters of the photomultiplier Δ_{T+D}, the diagram of the measurements is shown in Fig. 2b.

The results of measurements of the dependence of the error (relative error) on the number of photons accumulated[4] is presented in Fig. 1. The curves were obtained for the following cases: 1) the theoretical dependence $\Delta = f(N_p)$; 2) the contribution of the instability of the time intervals; 3) the sum of the instability of the time intervals and the parameters of the one-electron pulse discriminator. Figure 2 shows a diagram of the experiment for obtaining curve 2 (a) and curve 3 (b). One can see from Fig. 1 that the curves 2 and 3 reach a limit, whose magnitude does not depend on N. This limit can be interpreted as a measure of the estimate of the instrumental error of the measuring system in accordance with Eq. (6).

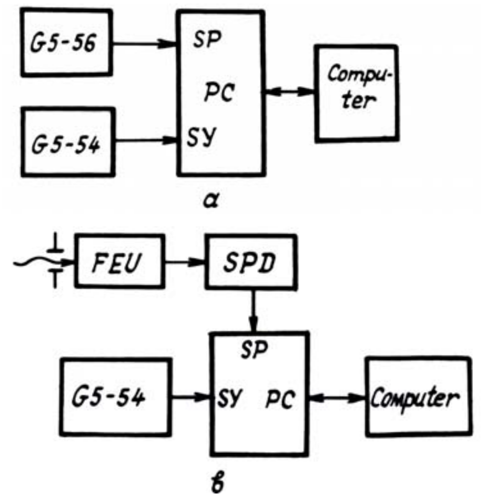

FIG. 2. Diagram of experiment: a) for measuring the contribution of fluctuations of the durations of the time intervals. G5–56 *— simulator of a flux of single-electron pulses distributed statistically in time relative to a synchronous pulse; b) for measuring the contribution of fluctuations of the discrimination thresholds and parameters of the photomultiplier. The intensity of the light flux from a thermal source on the photomultiplier is regulated by neutral filters. SP — signal input (single-electron pulses); SY — synchronization input; PC — photon counter; SPD — single-electron pulse discriminator.*

Thus taking into account the fluctuations of the parameters of the measuring apparatus, in particular, the fluctuations of the duration of the measuring intervals, gives a quite simple explanation of the experimentally observed behavior of the relative error when the accumulation volume is increased, and it makes it possible to obtain a value for this error. The proposed method is used in measuring the error in the detection of a flux of single-electron pulses in photon counters for laser sounding of the atmosphere.

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