RADIAL-FREQUENCY REPRESENTATION OF THE FIELD IN THE CROSS SECTION OF A SEARCHLIGHT TUBULAR BEAM FORMED WITH AN AXICON

O.I. Aldoshina and A.V. Fabrikov

Scientific Center for Optical-Physical Measurements, Moscow
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The equation is derived for the field in the cross section of a searchlight tubular beam formed with an axicon with a laser source at the input. The equation in the form of a Fourier integral transforms the function of radial coordinate at the axicon input into the function of radial frequency at its output in the Fraunhofer zone. The calculations made with the use of a fast Fourier algorithm permit the real time control over the searchlight beam shape at an arbitrary distance from the source based on data measured in the near zone. In a particular case of a homogeneous plane wave and Gaussian beam at the output, the integral can be calculated explicitly, and the solution may be represented as the degenerate hypergeometric Kummer functions.

By the radial-frequency representation of an axially symmetric function $\psi(r)$, $r = \sqrt{x^2 + y^2}$ we understand its representation with a one-side Fourier transform of some other axially symmetric function $\varphi(r')$, $r' = \sqrt{x'^2 + y'^2}$ of the frequency variable Δ dependent on r:

$$\psi(r) = \int\limits_0^\infty \varphi(r') \exp(i\,\Delta\, r')\, \mathrm{d}r' , \quad \Delta = \Delta(r) . \tag{1}$$

This representation is found to be very convenient for description of a searchlight tubular beam formed in the far zone by a system of a laser and axicon (or reflaxicon) as a linear spatial modulator of a coherent optical radiation phase.[1–3]

If a coherent optical radiation wave with the complex amplitude $\varphi_0(r')$, $r' = \sqrt{x'^2 + y'^2}$ is incident on an axicon located in the plane $z = z'$, at its output it will be described by the expression:

$$\varphi(r') = \varphi_0(r') \exp(i\,\omega_0\, r') , \tag{2}$$

where $\exp(i\omega_0 r')$ is the axicon transmission function; ω_0 is the parameter connected with the radiation wavelength λ and the angle of a beam deviation by the axicon, β, by the relation:

$$\omega_0 = k\beta , \quad k = 2\pi/\lambda . \tag{3}$$

In the case of a conic axicon with the angle α at the base ($\alpha \ll 1$) made from a dielectric material with the index of refraction n, the deviation angle β is equal to $(n - 1)\alpha$. For the reflaxicon with the angle α between the inner and outer reflecting surfaces, $\beta = 2\alpha$. In the paraxial Fraunhofer approximation of the scalar Kirchhoff diffraction theory the field $\psi(r)$ is defined by the Fourier-Bessel transform of $\varphi(r')$:

$$\psi(r) = \frac{k}{z} \int\limits_0^\infty \varphi(r')\, J_0(\omega\, r')\, r'\, \mathrm{d}r' , \tag{4}$$

where $J_0(\cdot)$ is the Bessel function of the first kind of zeroth order.[4] When the system has a collecting lens with a focal length f the distribution of complex amplitude ψ in the focal plane of the lens is described by the same equation at $z = f$.

A solution in the closed form was found in the approximation of large parameter $x = \omega_0 a$, where $2a$ is the cross size of a beam passed through the axicon (values of this quantity in actual practice is close to 10^3) in the case of a plane wave and a Gaussian beam at the axicon input.[5–7] In the first case $\varphi_0(r') = \mathrm{circ}(r'/a)$ is a circular function and

$$\psi(r) = \exp(-2\,i\,x) \sqrt{\frac{i\,4\,a^3}{9\,\lambda\, z\, r_0}}\; {}_1F_1(3/2, 5/2; -i\,2\,\rho) ,$$
$$r_0 = \beta z , \quad \rho = ka(r - r_0)/2z ; \tag{5}$$

${}_1F_1$ denotes the degenerate hypergeometric Kummer function. The derived from Eq. 5 formula for the radiation intensity $I(r) = |\psi(r)|^2$ can be approximated by the expression[8]:

$$I(r) \approx \frac{a^3}{2\,\lambda\, z\, r_0} \left| \frac{\sin\rho}{\rho} \right|^2 . \tag{5'}$$

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In the second case $\varphi_0(r') = \exp(-r'^2/a^2)$ and

$$\psi(r) = B\left[{}_1F_1(3/4,\ 1/2;\ -\rho^2) -\right.$$

$$\left.- 2\,i\,\rho\,\frac{\Gamma(5/4)}{\Gamma(3/4)}\,{}_1F_1(5/4,\ 3/2;\ -\rho^2)\right], \qquad (6)$$

$$B = \frac{\Gamma(3/4)}{2}\sqrt{\frac{a^3}{i\,\lambda\,z\,r_0}}\ ,$$

$\Gamma(.)$ is the gamma function. The intensity $I(r)$ is approximated by the expression

$$I(r) = 0.24\ \pi\ \frac{a^3}{2\,\lambda\,z\,r_0}\exp[-2(\rho/1.65)^2]\ . \qquad (7)$$

The middle radius of illumination circle r_0 and the circle width δ (at the intensity level e^{-2} of the maximum value) in the plane transverse to the beam at the distance z from the axicon are defined by the expressions:

$$r = z\,\beta\ ,\quad \delta = 3.3\,\frac{\lambda z}{\pi a} \approx \frac{\lambda z}{a}\ . \qquad (7')$$

In the general case it is impossible to obtain simple analytical expressions for $\psi(r)$ and $I(r)$ based on Eq. (4). But in the same approximation on the parameter $\omega_0 a$ one can reduce Eq. (4) to a form (1) convenient for numerical estimations. Let us use the asymptotic expansion of the function $J_0(y)$ (Ref. 9, p. 185):

$$J_0(y) = \sqrt{2/(\pi y)}\ \{\cos(y - \pi/4) + O(|y|^{-1})\}\ .$$

In our case $y = \omega r' = \omega a(r'/a)$, where ωa is value of the order of 10^3, and if the inequality $(r'/a) \geq 10^{-2}$ is fulfilled the rest term can be neglected. In the initial part of the integration domain $0 \leq r' < 10^{-2}a$ the inequality is not fulfilled, but it is inessential because of the factor r' in the integral expression. Substituting the asymptotic representation $J_0(y)$ into Eq. (4), omitting the remainder term $O(|\omega r'|^2)$, and taking into account Eq. (2), we obtain

$$\psi(r) = 2C\int\limits_0^\infty \varphi(r')\cos(\omega r' - \pi/4)\ \mathrm{d}r' =$$

$$= C\int\limits_0^\infty \varphi_0(r')\ \{\exp[i\,(\omega\,r' - \pi/4)] +$$

$$+ \exp[-i\,(\omega_0\,r' - \pi/4)]\}\ \exp(i\,\omega_0\,r')\ \mathrm{d}r'\ ,$$

$$C = \sqrt{2\pi/(\lambda^2\,z^2\,\omega)}\ .$$

The term with the rapidly oscillating factor $\exp[i(\omega_0 + \omega)r']$ yields, as a result of integration, the value very close to zero. Omitting it we obtain the equation of the form of Eq. (1)

$$\psi(r) = \int\limits_0^\infty \tilde{\varphi}_0(r')\ \exp(i\,\Delta\,r')\ \mathrm{d}r'\ , \qquad (8)$$

where

$$\tilde{\varphi}_0(r') = C\ \varphi_0(r')\sqrt{r'}\ ,$$

$$\Delta = \omega_0 - \omega = \frac{2\pi}{\lambda z}\,(r_0 - r)\ , \quad (9)$$

$$C \approx \sqrt{2\pi/(\lambda^2\,z^2\,\omega_0)} = 1/\sqrt{\lambda\,z\,r_0}\ .$$

We have omitted here the insignificant phase factor $\exp(i\pi/4)$ and take $\omega \approx \omega_0$. This is justified by the fact that the function $\psi(r)$ has a maximum at $r = r_0$ and rapidly falls off in magnitude at a distance r from r_0. Physically this means that the field of the wave passed through the axicon in the Fraunhofer zone is concentrated in a narrow layer (with the width δ) around a conic surface with the angle $r_0/z \approx \beta$ at vortex. It is the property of an axicon that is used in laser beacons with an searchlight beam shaped as a tubular cylinder with the circular distribution of radiation energy in every cross section.[1]

The wave transformed according to Eqs. (4) or (8) is usually plane but not always is uniform. The dependence $\varphi_0(r')$ can be too complicated, particularly in the case of a reflaxicon, and can change in time. The use of equation (8) in calculations makes it possible to permanently control the beam shape $I(r) = |\psi(r)|^2$ in the far zone using the data of measurements of $I_0(r')$ in the immediate vicinity of the axicon. Calculation of the function $\psi(r)$ on the base of the finite set of values $\tilde{\varphi}_0(r') = C\sqrt{r'\,I(r')}$ is performed with the use of a discrete linear filter realizing fast Fourier transform algorithm. The text of the corresponding PASCAL program is given in Appendix.

The equations (5) and (6) were obtained by approximation of infinite series, representing the right-hand side of Eq. (4) for a plane wave and a Gaussian beam, respectively. They can be considered as approximate solutions of the exact equation (4). But they are also exact solutions of the approximate equation (8). One can check this using formulas of the integral representation of the degenerate hypergeometric and parabolic cylinder functions. So the formula [Ref. 9, (13.2.1)]

$$\frac{\Gamma(b - a)\ \Gamma(a)}{\Gamma(b)}\ {}_1F_1(a,\ b;\ z) =$$

$$= \int_0^1 \exp(z\,t)\; t^{a-1}\; (1-t)^{b-a-1}\;\mathrm{d}t$$

at $z = i\rho$, $a = 3/2$ and $b = 5/2$ it is reduced, accurate to insignificant phase factors, to Eq. (5) and formula [Ref. 10, (7.341)]

$$D_p(z) = \frac{\exp(-z^2/4)}{\Gamma(-p)} \int_0^\infty \exp(-z\,x - x^2/2)\; x^{-p-1}\;\mathrm{d}x$$

because of the relationships existing between parabolic cylinder functions and Kummer functions [Ref. 10, (7.340)]

$$D_p(z) = 2^{p/2} \exp(-z^2/4) \times$$

$$\times \left\{ \frac{\sqrt{\pi}}{\Gamma[(1-p)/2]}\; {}_1F_1(-p/2,\, 1/2;\, z^2/2) - \right.$$

$$\left. - \frac{\sqrt{2\pi}\, z}{\Gamma(-p/2)}\; {}_1F_1[(1-p)/2,\, 3/2;\, z^2/2] \right\}$$

at $z = i\Delta a$ and $p = -3/2$ leads to Eq. (6).

In so doing, calculations by Eq. (8) in the case of a plane wave and a Gaussian beam give the same order of approximation as calculations by Eqs. (5) and (6) derived from Eq. (4). Its advantage in comparison with Eq. (4) is in it convenience for numerical calculations of functions $\varphi_0(r')$ of an arbitrary type. At fast time variations of $\varphi_0(r)$, tracked with a beam shape recorder, calculations of $\psi(r)$ by Eq. (8) can be performed in real time.

APPENDIX

The program for calculating discrete linear filter performing fast Fourier transform algorithm

```
Program IIi;
Type Real Array=Array[0...31] of Real;
Function Ibitr (j,nu:Integer):Integer;
Var i, j1,j2,k: Integer;
Begin
        j1:=j;
        k:=0;
        For i:=1 to nu do
    Begin
        j2:=j1 Div 2;
        k:=k*2+(j1−2*j2);
        j1:=j2
    End;
    Ibitr:=k
End; {ibitr}
Procedure FFT(VarXReal,XImag:RealArray;N,nu:Integer);
Var N2,Nu1,i,l,k,m: Integer;
TReal, TImag,p,arg,c,s: Real;
Label LBL;
Begin
N2:=N DIV 2;
NU1:=NU−1;
K:=0;
FOR L:=1 TO NU DO
BEGIN
    LBL:
        FOR I:=1 TO N2 DO
        BEGIN
        M:=K DIV ROUND(EXP(NU1 * LN(2)));
        P:=IBITR(M,NU);
        ARG:=6.283185*P/N;
        C:=COS(ARG);
        S:=SIN(ARG);
        TREAL:=XREAL[K+N2]*C+XIMAG[K+N2]*S;
        TIMAG:=XIMAG[K+N2]*C−XREAL[K+N2]*S;
        XREAL[K+N2]:=XREAL[K]−TREAL;
        XIMAG[K+N2]:=XIMAG[K]−TIMAG;
        XREAL[K]:=XREAL[K]+TREAL;
        XIMAG[K]:=XIMAG[K]+TIMAG;
                END;
                K:=K+N2;
                IF K<N THEN GO TO LBL;
                K:=0;
                NU1:=NU1−1;
                N2:=N2 DIV 2
END;
FOR K:=0 TO N−1 DO
BEGIN I:=IBITR(K,NU);
                IF I>K THEN
                BEGIN;
                        TREAL:=XREAL[K];
                        TIMAG:=XIMAG[K];
                        XREAL[K]:=XREAL[I];
                        XIMAG[K]:=TIMAG[I];
                        XREAL[I]:=TREAL;
                        XIMAGE[I]:=TIMAG;
END  END  END; {FFT}
BEGIN FFT(  ); END.
```

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