SIMULATING SOLAR RADIATION TRANSFER IN THE EARTH'S SPHERICAL ATMOSPHERE AND CLOUDS

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Received October 19, 1998
Accepted November 10, 1998

Radiation transfer is considered in a spherical axially symmetric layer. A boundary-value problem for the kinetic equation is solved by the method of successive approximations with the integration over characteristics of the partial differential operator.

INTRODUCTION

Recently the attention to the problem of radiation transfer in the atmosphere has strongly increased as a result of complex analysis of the physical, chemical, meteorological, and biological processes responsible for formation of the Earth's radiation field. Radiation processes play the leading part in the atmospheric heat and power exchange and, consequently, in climate formation on the global and local scales. The impact of anthropogenic and natural factors on the radiative processes in the atmosphere–Earth system (AES) may cause the destruction of the Earth's biosphere self-restoration potential thus leading to catastrophic consequences. That is why these problems attract not only pure scientific attention, but the public attention as well; they are nowadays discussed all over the world by a number of governmental and other organizations. Adequate understanding of the radiative processes is necessary for providing scientific and technological progress and for preventing possible negative consequences of the climate changes or significant deviations of the spectral-radiative balance of the planet. Unfortunately, no possible climate and biophysical changes can be predicted at present for certain. This is so, in particular, because of low accuracy of the description of radiation in the climate models as well as of the atmospheric and oceanic circulation.

A system of optical sensing of the atmosphere and the Earth's surface from ground-based stations, aircrafts, helicopters, air balloons, rockets, satellites, and orbiting manned stations could provide for quite an extended information. However, such data have a common drawback because it is impossible, under field conditions, to determine and adequately monitor, at a time, all the medium parameters, that are responsible for the formation of radiation field during the measurements. The situation is complicated by the fact that the medium continuously varies in time and space, thus preventing reproduction of the measurement conditions. The interpretation of experimental data together with theoretical calculations, made for the controllable "determinate" conditions, allows significant extension of our knowledge about the environment.

Constructing of radiation models of the Earth as a planet and the human habitat is of extreme importance for solution of a number of complicated applied and technical problems. Among such problems there are, in particular, the development of space-based systems for monitoring the Earth's surface; orientation, stabilization, navigation, and additional power supply of a spacecraft by using the back surface of solar batteries for reception of solar radiation reflected by the AES.

This paper continues our long-term research on the development of methods and algorithms for numerical solution of the problems of radiative transfer in scattering, absorbing, and emitting spherical systems with a complex structure.[1,2] This work has been stimulated, to a great extent, by a significant change in the information technologies due to implementation of high-efficiency multiprocessor computers with parallel structures.

The studies based on spherical multidimensional AES models have been started in the 60s in connection with first space flights and the development of astrophysics and planetary physics. Axially symmetric systems occupy a significant place in the theory of radiative transfer in 3D finite-volume media. Axially symmetric models sufficiently well reflect the basic features or mechanisms of physical processes in a number of problems. "esides, the up-to-date computers make the solution of such problems quite realistic. Spherical axially symmetric models are interesting in connection with the following problems:

0235-6880/99/03 240-07 $02.00 © 1999 Institute of Atmospheric Optics

1. Investigation of a radiation field in the atmosphere of a spherical planet illuminated by an external (e.g., solar) parallel radiation beam. This problem has various technical applications. At the same time, it is the classical problem of astrophysics and atmospheric optics;

2. Determination of the radiation field generated by a point source in an inhomogeneous spherical layer. It is not only applied, but also a classical problem of the radiative transfer theory, which involves calculation of the influence function (Green's function) of the boundary-value problem for the kinetic equation;

3. Examination of the reflection properties of a sphere illuminated by a parallel or diffuse external radiation beam. Such a sphere can serve as a model of an individual cumulus cloud or an optically dense particle of some turbid medium;

4. Investigation of the radiation field inside a spherical shell illuminated by an external radiation beam. This problem is taken from the theory of spacecraft protection against the radiation.

Introduction of an axial symmetry is a model element, which, on the one hand, only slightly distorts the physical process, while being, on the other hand, only a technical restriction (because of the multidimensional character of the problem), rather than the principle one (in order to make the problem less cumbersome). For example, such a model allows one to study purely spherical effects observed in polar regions, on the terminator, in twilight, near the Earth's limb and horizon. "esides, it can be applied to the problem of radiative transfer along a latitude, and the phase curve of the Earth's brightness (fullness of the brightness disc, similar to the phases of the moon). Another important problem that can be solved using this model is the brightness of stars observed against the bright background of the Earth's atmosphere, including limb directions, when the stars are observed from space and a sighting line passes above the Earth's disk through its atmosphere. "esides, the stars (passive sources), the active sources (such as lasers) can be used in the projects, now very urgent, which study the tomography of the Earth's atmosphere.

We are interested in calculation of the AES brightness field simultaneously on the global scale. The atmosphere is considered as a spherical shell of the Earth. For the case of observation from outside the atmosphere (for example, from a space orbit), the solution at the observation point is merely the solution obtained for the upper boundary of the shell without radiation extinction (taking into account the problem geometry). The complexity of the problem geometry is mostly due to shadowing of a vast areas by the Earth's disc. Therefore, we have to study a spherical shell, with the reflecting upper concave surface and the "vacuum" lower one, as well as the transparent cylinder side surface.

First spherical models have been studied by V.V. Sobolev and I.N. Minin (Refs. 3–9) mostly in the single scattering approximation; multiple scattering was taken into account partially in the diffusion approximation for a plane layer. This approach, referred to as Sobolev method, was significantly improved by O.I. Smoktii (Refs. 10–14) and L.G. Titarchuk (Refs. 15 and 16). The single scattering approximation was also used by O.A. Avaste (Refs. 17–19). G.I. Marchuk, G.A. Mikhailov, M.A. Nazarliev, R.A. Darbinyan, and V.S. Antyufeev, significantly contributed into solution of the spherical problems. They have laid grounds for the Monte Carlo method in the atmospheric optics.[20–31] Simultaneously T.A. Sushkevich has developed the determinate approach to modeling the global radiation field of the Earth.[1,2,32–36] The methods from Refs. 37–40 were analyzed comparatively as applied to interpretation of the first data acquired from space (Refs. 10–12, 19–21, and 32–36), in particular, spectrophotometric measurements of the Earth's horizon and background, as well as photographs of "space sunrise and sunset."

The basic method used in our calculations of the spherical AES is the iteration method of characteristics (IMC). It is a combination of the method of integration of the transfer equation over characteristics and the method of successive approximations by the number of scattering events supplemented with the procedures improving a convergence of iterations. As follows from our experience, the algorithm for its optimal implementation should provide for parallel calculations.[41–48]

The algorithms that use the method of characteristics (with or without interpolation) for two-dimensional spherical axially symmetric system were first developed by T.A. Sushkevich (Ref. 1). Some particular cases (with significant restrictions imposed on the structure of the scattering and absorbing medium, as well as on the illumination and observation conditions) of integration of the transfer equation in single-scattering approximation are considered in some papers by O.A. Avaste and O.I. Smoktii. Later and up to date, when the spherical problem is being solved by the Monte Carlo method, the single-scattering approximation is realized by integrating over characteristics, which coincide with the ray trajectories.

Among foreign scientists, there were Sekera and Lenobl (the U.S.A.) who attempted to solve the spherical problem.[49] They proposed to use the method of successive approximations corresponding to series expansion of solution over a small parameter. They use the solution of the plane problem as the first approximation and the ratio of the effective height of the homogeneous atmosphere to the Earth's radius as the small parameter. Most foreign papers use the Monte Carlo method or approximate methods.[42,49]

MATHEMATICAL STATEMENT OF THE PROBLEM

Let us consider the problem of radiation transfer through the Earth's atmosphere. The atmosphere is treated as a spherical shell illuminated by the external parallel flux of radiation. Let the direction of the OZ axis passing through the Earth's center be opposite to the incident flux. The atmosphere–Earth system is considered as two-dimensional: radius vector $\mathbf{r}$ of any point $A(\mathbf{r})$ is determined by the distance $r = |\mathbf{r}|$ from the Earth's center and the polar angle ψ measured from the axis of symmetry OZ of the system; $y = \cos\psi$. The direction of propagation of a light beam $\mathbf{s}$ at the point $A(\mathbf{r})$ is described in the local system of spherical coordinates: by the zenith angle ϑ measured from $\mathbf{r}$ and the azimuth φ in the plane tangent to the sphere of the radius r at the point $A(\mathbf{r})$. The set of all points $A(\mathbf{r})$ of the spherical shell forms the open domain G with the lower G_{low} and upper G_{up} boundaries – spherical surfaces with the radii R_{low} and R_{up}. The vector field of all directions of light beams $\mathbf{s}(A)$ at every point $A(\mathbf{r})$ forms the set of $\Omega = \Omega^+ \cup \Omega^-$ – a unit sphere, where Ω^+ and Ω^- are the hemispheres of $\mathbf{s}$ directions with $\mu^+ \in [0, 1]$ and $\mu^- \in [-1, 0]$; $\mu = \cos\vartheta$. Let us introduce the sets $b \equiv G_{low} \times \Omega^+$ and $t \equiv G_{up} \times \Omega^-$.

The task is to find the intensity of the attenuated direct radiation coming from sources and the stationary field of intensity of a single and multiply scattered radiation in a scattering, absorbing, and emitting spherical shell G with the boundaries G_{low} and G_{up} or beyond G. The total intensity of radiation $\Phi(\mathbf{r}, \mathbf{s})$ at the point $A(\mathbf{r})$ in the $\mathbf{s}$ direction will be sought as a solution to the General "oundary-Value Problem (G"VP) of the radiation transfer theory[1,2,41,43]

$$\hat{K}\Phi = F^{in}, \quad \Phi\big|_t = F^t, \quad \Phi\big|_b = \varepsilon\hat{R}\Phi + F^b \qquad (1)$$

in the phase area $\Gamma \equiv G \times \Omega + G_{up} \times \Omega^- + G_{low} \times \Omega^+$ with the linear operators: the transfer operator

$$\hat{D} = \frac{\partial}{\partial s} + \sigma_{tot}(\mathbf{r}), \qquad (2)$$

$$\left.\frac{\partial\Phi}{\partial s}\right|_{\mathbf{r}} =$$

$$= \cos\vartheta\frac{\partial\Phi}{\partial r} + \frac{\sin\vartheta}{r}\left[\cos\varphi\frac{\partial\Phi}{\partial\psi} - \frac{\partial\Phi}{\partial\vartheta} - \cot\psi\sin\varphi\frac{\partial\Phi}{\partial\varphi}\right];$$

the collision integral as a function of the source

$$B(\mathbf{r}, \mathbf{s}) \equiv \hat{S}\Phi = \sigma_{sc}(\mathbf{r}) \int_{\Omega} \gamma(\mathbf{r}, \mathbf{s}, \mathbf{s}')\, \Phi(\mathbf{r}, \mathbf{s}')\, d\mathbf{s}',$$

$$d\mathbf{s}' = d\mu'\, d\varphi'; \qquad (3)$$

the integrodifferential operator $\hat{K} \equiv \hat{D} - \hat{S}$;
the operator of reflection

$$[\hat{R}\Phi](\mathbf{r}_{low}, \mathbf{s}) = \int_{\Omega^-} q(\mathbf{r}_{low}, \mathbf{s}, \mathbf{s}^-)\Phi(\mathbf{r}_{low}, \mathbf{s}^-)\, d\mathbf{s}^-, \quad \mathbf{s} \in \Omega^+;$$

$\sigma_{tot}(\mathbf{r})$ and $\sigma_{sc}(\mathbf{r})$ are the spatial distributions of the total cross section of radiation interaction with substance and the scattering cross section. The function F^{in} is the density of radiation sources inside the shell G; F^b and F^t are the radiation sources at the boundaries of the spherical shell defined for the rays $\mathbf{s}$ directed inward the shell G; the parameter $0 \le \varepsilon \le 1$ fixes the act of interaction with the boundary.

The boundary-value problem (1) is considered under natural restrictions, following from the physics of the process under study and the restrictions imposed on the coefficients, sources, and boundary conditions:

a) $\sigma_{tot}(\mathbf{r})$ and $\sigma_{sc}(\mathbf{r})$ are finite, piecewise continuous and piecewise differentiable functions;

b) $\gamma(\mathbf{r}, \mathbf{s}, \mathbf{s}')$ is a continuous function of the scattering angle $\chi = \arccos(\mathbf{s}\cdot\mathbf{s}')$, piecewise differentiable with respect to every variable;

c) operators $\hat{S}$ and $\hat{R}$ are uniformly bounded: $0 \le \hat{S}(1)$, $\hat{R}(1) \le 1$;

d) the media inside the shell G, on G_{low} and G_{up} are nonmultiplicative;

e) $F^{in}(\mathbf{r}, \mathbf{s})$, $F^t(\mathbf{r}_{up}, \mathbf{s}^-)$, and $F^b(\mathbf{r}_{low}, \mathbf{s}^+)$ are bounded, piecewise continuous or finite functions.

Note that we consider here the spherical shell G. The problem for a sphere (for example, for a spherical cloud) can be reduced to the problem for a spherical shell by using the boundary conditions with reflecting boundary G_{low} having an infinitesimal radius R_{low}. These conditions describe radiation propagation through the inner sphere bounded by G_{low}. If the inner region is a cavity, then the "shooting through" condition is imposed on G_{low}. If the inner sphere is a scattering or absorbing medium, then the "reflection condition" is imposed on G_{low}, which accounts for radiation extinction inside the inner sphere. Such approaches extend the applicability of the radiation transfer model under consideration. In particular, they allow a number of problems of astrophysics and planetary physics to be involved.

The constructing of a solution of G"VP (1) is based on analysis of properties of the functions Φ and B

(their continuity, differentiability, local properties). The continuity and differentiability of the source function B with respect to angular variables is determined to a great extent by the properties of the scattering phase function. "ecause the scattering phase function depends on the scalar product $\mathbf{s}\cdot\mathbf{s}'$, rather than on any particular direction separately, B is a locally smooth function with respect to $\mathbf{s}$ (while the function Φ is smooth on the average). The degree of B smoothness with respect to spatial variables is the same as for the functions σ_{sc}, γ, and Φ. Discontinuity of B is possible only on such surfaces, where the functions $\sigma_{tot}(\mathbf{r})$ and $\sigma_{sc}(\mathbf{r})$ are discontinuous. Along the trajectories of characteristics, the function $\hat{D}^{-1}B$ is smoother than Φ. Spatial derivatives of B have logarithmic singularities in the vicinities of points $\mathbf{r} \in G$ on the surfaces of discontinuity of the coefficients $\sigma_{tot}(\mathbf{r})$ and $\sigma_{sc}(\mathbf{r})$ (Ref. 50).

METHOD OF CHARACTERISTICS

A solution to the boundary-value problem of the stationary transfer equation is sought by the method of successive approximations. This method assumes the use of ordinary collisional iterations of different multiplicity or modified iterations involving the accelerating procedures.

To invert the differential operator of the transfer equation (2), we perform the integration over the characteristics at every iteration when calculating approximations of any order. Of principal importance for a wide use of the method of characteristics is inclusion of interpolation[51] and use of additive properties of exponents in the scheme of subdivision of calculations by sections along the characteristics. We fix the direction $\mathbf{s} \in \Omega$. Then we draw a straight line through the point $A(\mathbf{r}(0))$ in this direction so that the equation for this straight line could be written as

$$\mathbf{r}(\xi) = \mathbf{r}(0) - \xi\mathbf{s}\ , \quad -\infty < \xi < \infty\ , \tag{4}$$

where $D(\mathbf{r}(\xi))$ is the current point along the straight line; $A(\mathbf{r}(0))$ is a fixed point on the straight line, from which the shift $\xi = |AD|$ along the straight line is measured. Using such straight lines, we can transform the points of the domain G into the points (A, ξ) in a one-to-one manner. This procedure transforms the functions measurable in $G \times \Omega$ into the functions measurable on Ω along the straight lines with directions $\mathbf{s}$. The straight lines (4) being photon paths are the characteristics of the linear differential operator (2) (Refs. 1, 2, and 44–47). The transfer equation (2) written in an adequate form

$$\frac{\partial\Phi}{\partial\xi} + \sigma_{tot}\,(\mathbf{r} - \xi\mathbf{s})\ \Phi(\mathbf{r} - \xi\mathbf{s},\ \mathbf{s}) = E(\mathbf{r} - \xi\mathbf{s},\ \mathbf{s}) \tag{5}$$

with the known right-hand side can be solved explicitly by integration over a characteristic:

$$\Phi(\mathbf{r},\ \mathbf{s}) = \Phi(\mathbf{r} - \xi\mathbf{s},\ \mathbf{s})\ \exp\left[-\int\limits_0^{\xi} \sigma_{tot}\,(\mathbf{r} - \xi'\mathbf{s})\ d\xi'\right] +$$

$$+ \int\limits_0^{\xi} E(\mathbf{r} - \xi'\mathbf{s},\ \mathbf{s})\ \exp\left[-\int\limits_0^{\xi'} \sigma_{tot}\,(\mathbf{r} - \xi''\mathbf{s})\ d\xi''\right] d\xi'\ . \tag{6}$$

If the problem (5) involves nonzero boundary sources f, then it can be reduced to the problem with zero boundary conditions using a transformation of the following form:

$$\Phi^0(\mathbf{r},\ \mathbf{s}) = f(\mathbf{r} - \xi\mathbf{s})\ \exp\left[-\int\limits_0^{\xi} \sigma_{tot}\,(\mathbf{r} - \xi'\mathbf{s})\ d\xi'\right] \tag{7}$$

and presenting the solution as a sum $\Phi = \Phi^0 + \Phi_d$. The function Φ^0 corresponds to direct radiation from a source and has the same properties as f, but somewhat smoothed due to the exponential factor in Eq. (7). For the function Φ_d corresponding to the multiply scattered diffuse component, we have the problem with a constant term in the equation $F_1 = F^{in} + \hat{S}\Phi^0$. Let us separate out the term corresponding to the first scattering event $\Phi_1 = \hat{D}^{-1} F_1$, that is, present the total field as a superposition $\Phi_d = \Phi_1 + \Phi_{d-1}$. In the equation for Φ_{d-1}, the constant term has the form $F_{d-1} = \hat{S}\ \hat{D}^{-1} F_1$. Upon integration along the beam $\mathbf{s}$ (the action of the $\hat{D}^{-1}$ operator) and all directions of the unit sphere Ω (the action of the $\hat{S}$ operator), it turns out to be smoothed (as compared to F^{in}) both over spatial and angular variables. A discontinuity of the function F_1 with respect to angular variables results in a discontinuity of F_{d-1} with respect to $\mathbf{r}$. As the index of scattering multiplicity increases, this discontinuity becomes smoother. However, the discontinuities with respect to $\mathbf{r}$ in the coefficients σ_{tot}, σ_{sc}, and γ manifest itself in Φ and B at all iterations. Particular attention in the development of the numerical algorithm is paid to local properties of the solution, what allows one to increase the accuracy of solution and to describe the behavior of a solution in the vicinity of singular points.

It follows from Eq. (6) that differential properties of Φ are determined by the corresponding properties of the functions B, σ_{tot}, and F within G and by smoothness of the boundaries, which is characterized by the differential properties of $\xi(\mathbf{r},\ \mathbf{s})$. Spatial and angular derivatives of the function Φ exist and they are bounded. In the vicinity of tangent directions $\mathbf{s}^*$ (to lines or surfaces of discontinuity of the coefficients σ_{tot} and σ_{sc}), the derivatives have singularities of the form[1]:

$$\frac{\partial\Phi}{\partial r} \sim \frac{1}{\sqrt{|\mathbf{r} - \mathbf{r}^*|}}, \quad \frac{\partial\Phi}{\partial s} \sim \frac{1}{\sqrt{|\mathbf{s} - \mathbf{s}^*|}}.$$

The function Φ is absolutely continuous along the beams $\mathbf{s}$. On the set of points $\mathbf{r}$ and $\mathbf{s}$ other than $\mathbf{r}^*$, $\mathbf{s}^*$, it satisfies the Hölder continuity conditions:

$$|\Phi(\mathbf{r} + \Delta\mathbf{r}, \mathbf{s} + \Delta\mathbf{s}) - \Phi(\mathbf{r}, \mathbf{s})| \sim A|\Delta\mathbf{r}|^{1/2} \; |\Delta\mathbf{s}|^{1/2}.$$

The smoothness properties are taken into account at interpolation in the algorithm of integration over the characteristics.

INTEGRATION OVER A CHARACTERISTIC WITHOUT INTERPOLATION

The integration of the transfer equation over a characteristic without interpolation is performed along the entire path of the beam $\mathbf{s}$ from the calculated point $A(\mathbf{r})$ to the point at which the beam $\mathbf{s}$ enters the region G through the upper G_{up} or lower G_{low} boundaries. Toward this end, Eq. (6) is used, where $\Phi(\mathbf{r} - \xi\mathbf{s}, \mathbf{s}) = f(\mathbf{r} - \xi\mathbf{s}, \mathbf{s})$ and E are the sources at the boundary and inside the shell. The calculation of Φ by Eq. (6) with the known functions E and f is implemented in the following algorithm.

1. The boundary (G_{low} or G_{up}), through which the beam $\mathbf{s}$ enters the shell G, and the distance ξ from the point $A(\mathbf{r})$ to the entrance point $Q(\mathbf{r} - \xi\mathbf{s})$ are determined.

2. Coordinates of the point Q with the radius vector $\mathbf{r}(\xi) = \mathbf{r} - \xi\mathbf{s} = (r_\xi, \psi_\xi)$ are sought along with the angles $(\vartheta_\xi, \varphi_\xi)$, which describe the beam direction $\mathbf{s}$ in the local coordinate system of the point Q; $f(\mathbf{r} - \xi\mathbf{s}, \mathbf{s}) = f(r_\xi, \psi_\xi, \vartheta_\xi, \varphi_\xi)$ is calculated from the corresponding boundary conditions.

3. Coordinates of the point D with the radius vector $\mathbf{r}(\xi') = \mathbf{r} - \xi'\mathbf{s} = (r', \psi')$ at a distance ξ' from the point A are sought along with the angles ϑ' and φ', which describe the direction $\mathbf{s}$ in the local coordinate system toward the point D.

4. The coefficients $\sigma_{\text{tot}}(\mathbf{r} - \xi'\mathbf{s})$ are estimated at the point $D(r', \psi')$; the source intensity $E(\mathbf{r} - \xi'\mathbf{s}, \mathbf{s}) = f(r', \psi', \vartheta', \varphi')$ at the point $D(r', \psi')$ is calculated in the direction $\mathbf{s}' = (\vartheta', \varphi')$ with the coordinates (ϑ', φ') of the direction $\mathbf{s}$ in the local coordinate system with the origin at the point D.

5. Integrals are taken by the quadrature formulas with adaptive choice of the integration step $\Delta\xi$, taking into account structure of the coefficients $\sigma_{\text{tot}}(\mathbf{r})$, the source $E(\mathbf{r}, \mathbf{s})$, position of the point $A(\mathbf{r})$, the beam direction $\mathbf{s}$, and the scale of the section length ξ of the beam trajectory $\mathbf{s}$.

6. The function $\Phi(\mathbf{r}, \mathbf{s})$ is calculated by Eq. (6).

This algorithm allows one to choose the beam directions $\mathbf{s}$ independently and the spatial difference network – arbitrarily. The single-scattering approximation is the basic for spherical models, because it is just this approximation that is used when solving inverse problems in addition to the brightness field component reflecting all peculiarities of the problem. Calculations in the single-scattering approximation use the method of characteristics without interpolation. Such an approach is time consuming. However, the possibility to include any source and complex media into the algorithm is its important advantage. At the same time, the above-mentioned disadvantage is compensated for, to a great degree, by the algorithm enabling parallel calculations.

INTEGRATION OVER A CHARACTERISTIC WITH INTERPOLATION

The integration of the transfer equation over a characteristic with interpolation is used for calculation of the complete set of values of the difference functions $\Phi(\mathbf{r}_m, \mathbf{s}_n)$ with known difference functions $E(\mathbf{r}_m, \mathbf{s}_n)$. In this case, a difference network with respect to all derivatives is introduced in the regions G and Ω. The radii r_k and the polar angles ψ_l form the spatial difference network: $\mathbf{r}_m = (r_k, \psi_l)$. The set of calculated directions of the beam $\mathbf{s}_n$ at every point $\mathbf{r}_m$ of the layer is determined by a pair of angles $\mathbf{s}_n = (\vartheta_i, \varphi_j)$. The radiation intensity $\Phi(\mathbf{r}_m, \mathbf{s}_n)$ at the point $A(\mathbf{r}_m)$ in the direction $\mathbf{s}_n$ is calculated by taking integral of the equation

$$\frac{\partial\Phi}{\partial s_n} + \sigma_{\text{tot}}(\mathbf{r}_m)\,\Phi(\mathbf{r}_m, \mathbf{s}_n) = E(\mathbf{r}_m, \mathbf{s}_n)$$

over the characteristic – the beam $\mathbf{s}_n$:

$$\Phi(\mathbf{r}_m, \mathbf{s}_n) = \Phi(\mathbf{r}', \mathbf{s}'_n)\exp\left[-\int_0^{\xi} \sigma_{\text{tot}}(\mathbf{r}_m - \xi'\mathbf{s}_n)\,d\xi'\right] +$$

$$+ \int_0^{\xi} E(\mathbf{r}_m - \xi'\mathbf{s}_n, \mathbf{s}_n)\exp\left[-\int_0^{\xi'} \sigma_{\text{tot}}(\mathbf{r}_m - \xi''\mathbf{s}_n)\,d\xi''\right]d\xi'.$$

The shift $\xi = |\mathbf{r}_m - \mathbf{r}'|$ along the beam $\mathbf{s}_n$ is taken within a spatial cell. A cell is a ring bounded by two conic (y_1, y_2) and two spherical (r_1, r_2) belts. A beam can enter a cell either through the boundaries r_2 and r_1, or through the boundaries y_2 and y_1. We denote the coordinates of the entrance point as (r', ψ') and the direction $\mathbf{s}$ as (ϑ', φ'). To take the integral over the section $[0, \xi]$ of the characteristic, an interpolation by ξ is introduced for the function $E(\mathbf{r}, \mathbf{s})$ between its values at the nodes $E(r_1, \psi_1, \vartheta_1, \varphi_1)$ and $E(r', \psi', \vartheta', \varphi')$. The values $\Phi(r', \psi', \vartheta', \varphi')$ and $E(r', \psi', \vartheta', \varphi')$ at the arguments r', ψ', ϑ', and φ' other than nodes of the difference network are calculated by interpolation between neighboring nodes of the difference network.[1,2]

To construct the algorithm of integration over the characteristics, the equation of characteristics should be elaborated in details in order to provide for such a set of arguments r_k, ψ_l, ϑ_i, and φ_j, at which the iterations for obtaining values of $\Phi(r_k, \psi_l, \vartheta_i, \varphi_j)$ do not mix. "esides, it is necessary to be able to find the values of all four arguments r, ψ, ϑ, and φ at any point of the beam trajectory **s** from the given values of r_1, ψ_1, ϑ_1, and φ_1. It cannot be done based only on the geometry of the problem. The approach, which is based on the analysis of the analytical equation for a beam trajectory in the space of variables r and y with regard for the first integrals of the partial differential operator of transfer,[1,2,44–47] proves to be universal.

Calculating a single beam $\mathbf{s}_n = (\vartheta_i, \varphi_j)$ at the point $\mathbf{r}_m = (r_k, \psi_l)$ reduces to the following algorithm.

1. The point, where the beam $\mathbf{s}_n$ enters a cell, is calculated from the given values of r_k, ψ_l, ϑ_i, and φ_j and the coordinates r', ψ', ϑ', and φ' of the entrance point.

2. Values of $\Phi(r', \psi', \vartheta', \varphi')$ and $E(r', \psi', \vartheta', \varphi')$ are calculated by interpolation between neighboring values at the nodes of the difference network.

3. Integrals are calculated by the quadrature method with an adaptive step $\Delta\xi$, which takes into account, in particular, the structures of the coefficients $\sigma_{\text{tot}}(r, \psi)$.

The accuracy of quadrature equations used, when calculating the values of the source function $B(\mathbf{r}_m, \mathbf{s}_n)$, worsens at strong anisotropy of scattering. This effect may result in a divergence of iterations due to exceeding the unity norm of the operator $\hat{S}$ given by Eq. (3). In such situations, the approach involving a separation of strongly elongated part of the scattering phase function in the form of δ-function is used,[43] and the sequence of calculations differs from the ordinary iterations. To take into account the selective absorption in the multiple-scattering approximation, the method of subgroups with exponential approximation of the transmission function has been developed.[35]

ACKNOWLEDGMENTS

The work was supported in part by the Russian Foundation for "asic Research (Project No. 97–01–00995).

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