Accuracy and applicability domain of the small angle approximation

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Received August 26, 2004

The small angle approximation of the radiative transfer equation is one of the most efficient methods of solving applied problems of atmospheric and ocean optics. The analytical complexity of the radiative transfer equation (RTE) has allowed formulating only primary, qualitative estimates of the accuracy and applicability domain of the small angle approximation (SAA). Another approach based on the comparison of the SAA calculations of light fields with similar numerical RTE solutions is also possible. However, numerical methods of RTE solution for the cases of media with strong anisotropic scattering are inefficient. The approach proposed in this study can be readily generalized to the cases of arbitrary boundary conditions.

It is a peculiarity of formation of any natural turbid media that the size of suspended particles is much larger than the light wavelength. According to the Mie theory, this leads to the strongly anisotropic scattering. Difficulties of analytical and numerical solution of the boundary-value problems of the radiative transfer equation (RTE) in the case of anisotropic scattering initiated the development of specialized approximate methods, which altogether are referred to as the small angle approximation (SAA).[1]

The radiation transfer through a turbid medium, either atmosphere or seawater, is a particular case of transfer phenomena, described by the equivalent equations, but with the parameters of media having different physical meaning. Any process, characterized by the local compactness,[2] is described by the equation, fully analogous to RTE. Therefore, the methods for solution of both the transfer equations in general and RTE in particular are developed by specialists in various fields of physics.

Today SAA is one of the most powerful methods for solution of applied problems in atmospheric and ocean optics.[1] This causes the importance of considering its accuracy and applicability, as of any approximate method. However, analytical difficulties in RTE solution have allowed the formulation of any primary, qualitatively illustrative estimates.[1] Another approach is also possible here, namely, the comparison of light fields in a turbid medium, calculated in SAA, with the analogous numerical RTE solutions. However, most numerical methods also face serious difficulties in solving RTE for media with strongly anisotropic scattering, which does not allow the SAA accuracy to be estimated. In Ref. 3, we have proposed a numerical solution of RTE for turbid media with arbitrary scattering anisotropy under the exposure to radiation from a plane unidirectional (PU) source, which opens the possibility of solving the problem formulated.

Consider the boundary-value problem of RTE for the PU source at the arbitrary angle of irradiation of a semi-infinite layer of turbid medium

$$\mu\frac{\partial L(\tau,\mu,\varphi)}{\partial\tau} = -L(z,\mu,\varphi) + \frac{\Lambda}{4\pi}\oint L(z,\mu',\varphi')\,x(\hat{\mathbf{l}},\hat{\mathbf{l}}')\,d\hat{\mathbf{l}}', \qquad L(0,\mu,\varphi)\big|_{\mu>0} = \delta(\hat{\mathbf{l}}-\hat{\mathbf{l}}_0), \tag{1}$$

where $L(\tau,\mu,\varphi)\equiv L(\tau,\hat{l})$ is the brightness of the light field at the optical depth $\tau = \varepsilon z$ in the direction

$$\hat{l} = \{\sqrt{1-\mu^2}\cos\varphi;\ \sqrt{1-\mu^2}\sin\varphi;\ \mu\},\quad \mu = (\hat{l},\hat{z});$$

Λ is the single scattering albedo; ε is the extinction coefficient; $x(\hat{l},\hat{l}')$ is the scattering phase function of the medium;

$$\hat{l}_0 = \mu_0\hat{z} + l_{0\perp},\quad \mu_0 = (\hat{l}_0,\hat{z})$$

is the direction of radiation of the PU source. The axis OZ is perpendicular to the layer boundary; $\hat{z}$ is the unit vector in the direction OZ. The symbol "$\wedge$" denotes the unit vector.

SAA was first formulated in the studies of propagation of a flux of elementary particles through matter, which have led to the creation of a planetary model of an atom by Rutherford. The following problem appeared in this case: if the angular distribution of α-particles corresponded to the single scattering phase function of particles by a nucleus and confirmed the Rutherford model, then β-particles had the Gaussian angular distribution, which corresponded to the model of J.J. Thompson. In Ref. 4, it was supposed that β-particles experience multiple scattering events, and the equation was proposed for calculating scattering of arbitrary multiplicity in the problem (1), which can be written, in the terms of optics of turbid media, as follows:

$$L_n(\tau,\hat{l}) = \left(\frac{\Lambda}{4\pi}\right)^n \int\limits_0^{\xi_n}\ldots\int\limits_0^{\xi_1} \oint\ldots\oint \exp\left[-\xi-\sum_{k=1}^{n}\zeta_k\right] x(\hat{l},\hat{l}_n)\ \ldots$$

$$\ldots\ x(\hat{l}_2,\hat{l}_1)L_0(\hat{l}_1)\,d\hat{l}_1\ldots d\hat{l}_n\,d\zeta_1\ldots d\zeta_n. \tag{2}$$

0235-6880/05/01–02 32-06 $02.00

In the case of n scattering events, the ray trajectory is a broken line of $n+1$ segments. The optical length and the direction of each of n first segments from the PU source are denoted as ζ_i and $\hat{l}_i$, respectively. The length of the last segment, adjacent to the detector, is

$$\xi = \tau / \mu - \sum_{k=1}^{n} \zeta_k.$$

Since the scattering of electrons within the matter is strongly anisotropic, we can neglect, at small angles,[4] the variance of the path length of the scattered rays

$$\xi + \sum_{k=1}^{n} \zeta_k \approx \frac{\tau}{\mu_0}, \qquad (3)$$

which, after substitution into Eq. (2), allows Eq. (3) to be transformed as

$$L_n(\tau,\hat{\mathbf{l}}) = \frac{1}{n!}\left(\frac{\Lambda\tau}{4\pi\mu_0}\right)^n \exp\left(-\frac{\tau}{\mu_0}\right) \underbrace{\int \ldots \int}_{n} x(\hat{\mathbf{l}},\hat{\mathbf{l}}_1) \ldots$$

$$\ldots x(\hat{l}_{n-1},\hat{l}_n) L_0(\hat{l}_n) \mathrm{d}\hat{l}_1 \ldots \mathrm{d}\hat{l}_n.$$

Whence the full brightness of the light field is

$$L(\tau,\hat{\mathbf{l}}) = \sum_{n=0} L_n(\tau,\hat{\mathbf{l}}) =$$

$$= e^{-\tau/\mu_0} \sum_{n=0} \frac{(\Lambda\tau/\mu_0)^n}{n!} \left(\frac{1}{4\pi}\right)^n$$

$$\underbrace{\int \ldots \int}_{n} x(\hat{\mathbf{l}},\hat{\mathbf{l}}_n) \ldots x(\hat{\mathbf{l}}_2,\hat{\mathbf{l}}_1) \mathrm{d}\hat{\mathbf{l}}_1 \ldots \mathrm{d}\hat{\mathbf{l}}_n =$$

$$= e^{-\tau/\mu_0} \sum_{n=0} \frac{(\Lambda\tau/\mu_0)^n}{n!} \Phi_n(\hat{\mathbf{l}}). \qquad (4)$$

It can easily be seen that the small angle approximation is equivalent to the transformation of RTE (1) to the form

$$(\hat{l}_0, \nabla) L(\tau,\hat{l}) = -L(z,\hat{l}) + \frac{\Lambda}{4\pi} \int x(\hat{l},\hat{l}') L(\tau,\hat{l}') \mathrm{d}\hat{l}'. \qquad (5)$$

It is seen from Eq. (4) that the central problem in the small angle approximation is the calculation of the multidimensional convolution over the solid angle on the sphere $\Phi_n(\hat{l})$. Three main ways for solving this problem are possible.

In the historically first proposed approach,[5] it was postulated, based on the findings of the probability theory, that the multidimensional convolution of smooth functions tends to the normal distribution. This postulate was rigorously proved in Ref. 6. Let us present the field brightness in the direction $\hat{l}'$ by a Taylor series

$$L(z,\mathrm{r},\hat{l}') = \sum_{n=0}^{\infty} \frac{(\hat{l}'-\hat{l})^n}{n!} \nabla_{\mathrm{l}}^n L.$$

Then substitute this series into the scattering operator in RTE (5) with the first three terms kept on the assumption that the scattering phase function stronger depends on the angle than the brightness body. Thus, RTE (5) is reduced to the equation of the Fokker–Planck type (diffusion approximation):

$$(\hat{l}_0,\nabla) L(z,\hat{l}) + \kappa L(z,\hat{l}) = \frac{\langle \vartheta^2 \rangle}{2} \Delta_{\mathrm{l}} L, \qquad (6)$$

whose solution is a gaussoid in terms of the sighting angle. Here

$$\langle \vartheta^2 \rangle = \frac{1}{4\pi} \int x(\hat{\mathbf{l}}\hat{\mathbf{l}}')(\hat{\mathbf{l}} - \hat{\mathbf{l}}')^2 \mathrm{d}\hat{\mathbf{l}}' \approx \frac{1}{2} \int_0^{\pi} x(\cos\vartheta)\vartheta^2 \sin\vartheta \mathrm{d}\vartheta;$$

∇_{l}^n is the gradient in the space of sighting directions; $\kappa = (1 - \Lambda)\varepsilon$ is the absorption coefficient.

The additional assumption on the high anisotropy of the scattering phase function in comparison with the brightness body strongly smoothes all peculiarities of the solution and is valid only for small angles in the deep light regime. This fact has found the confirmation in the experiments with electron beams, and in Ref. 7 the authors tried to refine the solution to large angles, by "sewing" it with the single scattering approximation.

In Ref. 8, the second form of SAA was proposed, based on the summation theorem for Legendre polynomials. If $L_0(\mathrm{r},\hat{l})$ and $x(\hat{l}\cdot\hat{l}')$ are presented by the series expansion over of spherical functions:

$$L_0(\tau,\hat{l}) = \sum_{k=0}^{\infty} \frac{2k+1}{4\pi} \mathrm{P}_k(\hat{l}\cdot\hat{l}_0); \quad x(\hat{l},\hat{l}') = \sum_{k=0}^{\infty} (2k+1)\mathrm{P}_k(\hat{l}\cdot\hat{l}'), \qquad (7)$$

then for a PU source the equation for convolution takes the form

$$\Phi_n(\hat{l}) = \sum_{k=0}^{\infty} \frac{2k+1}{4\pi} x_k^n \mathrm{P}_k(\hat{l}\cdot\hat{l}_0). \qquad (8)$$

Whence

$$L(\tau,\hat{l}) = e^{-\tau/\mu_0} \sum_{k=0}^{\infty} \frac{2k+1}{4\pi} \exp\left(\frac{\Lambda\tau x_k}{\mu_0}\right) P_k(\hat{l}\cdot\hat{l}'). \qquad (9)$$

The third form of SAA was proposed in Refs. 9–12. In this form, along with the neglect of the path variance, it is assumed that, within small angles, the convolution on a sphere (rotation) can be replaced by the convolution on a plane, tangent to the sphere at $\hat{l}_0$ (plane shift):

$$\Phi_n(\hat{l}) \approx$$

$$\approx \underbrace{\int_{-\infty}^{+\infty} \ldots \int_{-\infty}^{+\infty}}_{n} x(\mathrm{l}_\perp - \mathrm{l}_{\perp n}) \times \ldots \times x(\mathrm{l}_{\perp 2} - \mathrm{l}_{\perp 1}) L_0(z,\mathrm{l}_{\perp 1}) \mathrm{dl}_{\perp 1} \ldots \mathrm{dl}_{\perp n}, \qquad (10)$$

where $\mathrm{l}_\perp$ is the projection of $\hat{l}$ onto the plane, tangent to the sphere at the point $\hat{l}_0$.

Upon the transition to Fourier transforms of $L_0(z,\hat{l})$ and $x(l_\perp - l'_\perp)$, according to the convolution theorem for the case of a PU source, we obtain

$$L(\tau,l_\perp) = e^{-\tau/\mu_0}\int_{-\infty}^{+\infty} L_0(k)\exp\left(\frac{\Lambda\tau x(k)}{\mu_0}\right)J_0(kl_\perp)k\mathrm{d}k, \quad (11)$$

where $J_0(\cdot)$ is the first-kind zero-order Bessel function.

In Refs. 13 and 14, the relationship between the three forms of SAA was considered and their accuracy was compared. It was shown that the second form is the most consistent and accurate, because it neglects only the path variance of scattered photons, but does not allow the extension to other sources, because $L_0(\hat{l})$ is independent of ξ due to the symmetry of the PU source. The third form follows from the second one with the restriction to small angles, when rotations on a sphere can be replaced by transformation of a plane:

$$P_k(\cos\vartheta) \approx J_0(k\vartheta), \quad \sum_{k=0}^{\infty}\frac{2k+1}{4\pi} \to \frac{1}{2\pi}\int_{-\infty}^{+\infty} k\mathrm{d}k. \quad (12)$$

The transition from the second and third SAA forms to the first one is based on the expansion of x_k or $x(k)$ into the Taylor series with the first three terms kept. It should be noted that the representation (3) is inconsistent with the exact boundary conditions,[14] and SAA neglects backscattering in the boundary conditions.

Though the most complete SAA form is the second form,[8] as all other forms follow from it, the analytical form of the third SAA form[10–12] appeared to be the most fruitful for solution of many applied problems.[1] In Ref. 15, this form of SAA was extended to the vector case of accounting for the particle spin, which is equivalent to the account for the transfer of polarized radiation in optics. However, the extension of the second SAA form is possible, which allows its limitations to be lifted.

In the boundary-value problem (1), let us pass on to the system of differential equations of the method of spherical harmonics,[3] but in the system with respect to the direction of radiation incidence $\hat{l}_0$. To do this, represent the solution in the form

$$L(\tau,\mu,\phi) = \sum_{k=0}^{\infty}\sum_{m=-\infty}^{\infty}\frac{2k+1}{4\pi}\sqrt{\frac{(k-m)!}{(k+m)!}}C_k^m(\tau)P_k^m(\nu)e^{im\varphi} =$$

$$= \sum_{k=0}^{\infty}\sum_{m=-\infty}^{\infty}\frac{2k+1}{4\pi}C_k^m(\tau)Q_k^m(\nu)e^{im\varphi}, \quad (13)$$

where

$$\nu = (\hat{l},\hat{l}_0); \quad \mu = \nu\mu_0 + \sqrt{1-\nu^2}\sqrt{1-\mu_0^2}\cos\varphi;$$

$P_l^n(\mu)$ are the associated Legendre polynomials, and $P_l(\mu) \equiv P_l^0(\mu)$;

$$Q_l^n(\mu) = \sqrt{\frac{(l-n)!}{(l+n)!}}P_l^n(\mu)$$

are the re-normalized Legendre polynomials.[3]

Substitute Eq. (13) into Eq. (1), multiply it by $Q_k^m(\nu)e^{im\varphi}$, and integrate over the total solid angle. Taking into account the orthogonality, recurrence relations, and the summation theorem for the Legendre polynomials, we obtain a system of connected differential equations:

$$\mu_0\frac{\mathrm{d}}{\mathrm{d}\tau}\Big[\sqrt{(k-m)(k+m)}C_{k-1}^m(\tau) +$$
$$+\sqrt{(k+m+1)(k-m+1)}C_{k+1}^m(\tau)\Big] +$$
$$+\frac{\sqrt{1-\mu_0^2}}{2}\frac{\mathrm{d}}{\mathrm{d}\tau}\Big\{\Big[\sqrt{(k-m+1)(k-m+2)}C_{k+1}^{m-1}(\tau) -$$
$$-\sqrt{(k+m)(k+m-1)}C_{k-1}^{m-1}(\tau)\Big] +$$
$$+\Big[\sqrt{(k+m+1)(k+m+2)}C_{k+1}^{m+1}(\tau) -$$
$$-\sqrt{(k-m-1)(k-m)}C_{k-1}^{m+1}(\tau)\Big]\Big\} =$$
$$= -(2k+1)(1-\Lambda x_k)C_k^m(\tau). \quad (14)$$

The system obtained differs from that commonly accepted in the method of spherical harmonics (SH) for a PU source[3] by a different coordinate system in the space of directions of sighting of the brightness referred to the axis $\hat{l}_0$, instead of $\hat{z}$.

For the further solution, let us take the following assumptions based on the concept of strong anisotropy of the brightness body.

1) Introduce a continuous dependence of the coefficients of the series (13) on the numbers of harmonics $C_k^m(\tau) = C^m(\tau,k)$, which is a slow monotonic function due to the angular anisotropy. Thus, the following representation is possible:

$$C^m(\tau,k\pm 1) \approx C^m(\tau,k) \pm \frac{\partial C^m(\tau,k)}{\partial k}. \quad (15)$$

2) The main contributors to the series (13) are the terms with the numbers $k \gg 1$, and the degree of anisotropy is much higher than its azimuth asymmetry $k \gg m$, which allows one to assume that:

$$\sqrt{(k-m)(k+m)} \approx k;$$

$$\sqrt{(k-m+1)(k-m+2)} \approx \kappa - \frac{(m-1)(2k+1)}{2\kappa};$$

$$\sqrt{(k+m+1)(k+m+2)} \approx \kappa + \frac{(m+1)(2k+1)}{2\kappa};$$

$$\sqrt{(k+m+1)(k-m+1)} \approx k+1;$$

$$\sqrt{(k+m)(k+m-1)} \approx \kappa + \frac{(m-1)(2k+1)}{2\kappa};$$

$$\sqrt{(k-m)(k-m-1)} \approx \kappa - \frac{(m+1)(2k+1)}{2\kappa},$$

where the following designation is introduced: $\kappa = \sqrt{k(k+1)}$.

Substituting these expressions into Eq. (14) and replacing k with κ, we obtain

$$\mu_0 \frac{\partial C^m}{\partial \tau} + \frac{\sqrt{1-\mu_0^2}}{2} \frac{\partial}{\partial \tau} \times$$

$$\times \left[\frac{\partial C^{m+1}}{\partial k} + \frac{\partial C^{m-1}}{\partial k} + \frac{1}{\kappa}\Big((m+1)C^m - (m-1)C^m\Big) \right] =$$

$$= -(1 - \Lambda x_k) C^m(\tau, \kappa). \qquad (16)$$

In Eq. (16) with regard for the condition $k \gg 1$ and the assumption of the slow monotonic dependence of C^m on the k number, the term with the second derivative is rejected as a small parameter. From the form of Eq. (16), it is seen that, unlike the SAA,[8] in which the system of equations of the SH method "breaks out" into independent equations for each of the harmonic coefficients, here the equations are interconnected, which is more adequate to the process of radiative transfer in a medium.

Introduce the function $f(\tau, \mathrm{l}_{0\perp}, \kappa)$ so that its azimuth spectrum is equal to the function sought:

$$f(\tau, \mathrm{l}_{0\perp}, \kappa) = \sum_{m=-\infty}^{\infty} C^m(\tau, \kappa) e^{im\psi}, \qquad (17)$$

and

$$C^m(\tau, \kappa) = \frac{1}{2\pi} \int_0^{2\pi} f(\tau, \mathrm{l}_{0\perp}, \kappa) e^{-im\psi} d\psi.$$

The following properties of the series (17) can be easily noticed:

$$\sum_{m=-\infty}^{\infty} C^{m\pm1}(\tau, \kappa) e^{im\psi} = e^{\mp i\psi} f(\tau, \mathrm{l}_{0\perp}, \kappa),$$

$$\sum_{m=-\infty}^{\infty} (m \pm 1) C^{m\pm1}(\tau, \kappa) e^{im\psi} = -ie^{\mp i\psi} \frac{\partial f(\tau, \mathrm{l}_{0\perp}, \kappa)}{\psi}. \qquad (18)$$

Multiply Eq. (16) by $e^{im\psi}$ and sum up over m from $-\infty$ to $+\infty$. Then, based on the properties (18), we obtain the equation

$$\sum_{m=-\infty}^{\infty} \left[\frac{\partial C^{m+1}}{\partial \kappa} + \frac{\partial C^{m-1}}{\partial \kappa} + \frac{1}{\kappa}\Big((m+1)C^m - (m-1)C^m\Big) \right] e^{im\psi} =$$

$$= 2\left(\cos\psi \frac{\partial f}{\partial \kappa} - \frac{\sin\psi}{\kappa} \frac{\partial f}{\partial \psi} \right) = 2(\hat{\mathrm{l}}_{0\perp}, \nabla_\kappa) f,$$

which makes the Eq. (16) to read

$$\frac{\partial}{\partial \tau}\Big[\mu_0 f + (\mathrm{l}_{0\perp}, \nabla_\kappa) f\Big] = -(1 - \Lambda x_k) f(\tau, \mathrm{l}_{0\perp}, \kappa). \qquad (19)$$

Let us present the solution of Eq. (19) in the form

$$f(\tau, \mathrm{l}_{0\perp}, \kappa) = \exp\left[-\frac{\tau}{\mu_0}\Big(1 - \Lambda F(\mathrm{l}_{0\perp}, \kappa)\Big) \right], \qquad (20)$$

which allows Eq. (19) to be solved by integration along the characteristic

$$F(\mathrm{l}_{0\perp}, \kappa) =$$

$$= \frac{\mu_0}{\sqrt{1-\mu_0^2}} \int_{-\infty}^{0} x\Big(\big|\kappa + \zeta \hat{\mathrm{l}}_{0\perp}\big|\Big) \exp\left(\frac{\mu_0}{\sqrt{1-\mu_0^2}} \zeta \right) d\zeta + u_0(z), \qquad (21)$$

where u_0 is an arbitrary function, determined from the boundary conditions.

The boundary conditions in the boundary-value problem (1) include the δ-function in terms of the sighting angle, which means $F(z, \mathrm{l}_{0\perp}, \kappa) \xrightarrow[\kappa \to 0]{} 0$. Whence $u_0 = 0$, and the solution of Eq. (19) finally takes the form

$$f(\tau, \mathrm{l}_{0\perp}, \kappa) = \exp\left\{ -\frac{\tau}{\mu_0} + \frac{\Lambda\tau}{\mu_0} \int_0^{\infty} x\left(\left| \kappa - \zeta \frac{\sqrt{1-\mu_0^2}}{\mu_0} \hat{\mathrm{l}}_{0\perp} \right| \right) e^{-\zeta} d\zeta \right\}. \qquad (22)$$

Correspondingly, for the coefficients of harmonics we obtain

$$C_k^m(\tau) = \exp\left(-\frac{\tau}{\mu_0} \right) \frac{1}{2\pi} \times$$

$$\times \int_0^{2\pi} \cos m\varphi \exp\left[\frac{\Lambda\tau}{\mu_0} \int_0^{\infty} x\left(\sqrt{\kappa^2 + a^2\zeta^2 - 2\kappa a \zeta \cos\varphi} \right) e^{-\zeta} d\zeta \right] d\varphi, \qquad (23)$$

where

$$a = \tan\theta_0 = \sqrt{1-\mu_0^2} \big/ \mu_0.$$

The Henyey–Greenstein scattering phase function is often used in applications. In this case, $x_k = g^k$ and the internal integral in Eq. (23) takes the form

$$\int_0^{\infty} x\left(\sqrt{\kappa^2 + a^2\zeta^2 - 2\kappa a \zeta \cos\varphi} \right) e^{-\zeta} d\zeta =$$

$$= \frac{\kappa}{a} \int_0^{\infty} \exp\left[\frac{\kappa}{a}\Big(a \ln g \sqrt{1 + \zeta(\zeta - 2\cos\varphi)} - \zeta \Big) \right] d\zeta.$$

The solution obtained is a small-angle modification of the method of spherical harmonics (MSH) and an extension of the second form of SAA, into which it transforms at $\mu_0 \to 1$. The MSH can be easily extended to the cases of an arbitrary radiation source.[16–18] The analytical form of MSH as a series in terms of spherical functions is much more convenient in numerical calculations, than the calculation of the improper integral in the SAA form.[9–12] The scattering phase function of the medium enters into the solution in the form of the coefficients of expansion in terms of the Legendre polynomials, which allows one to directly use their values from the model of optical characteristics of the medium and the Mie theory[19] without additional approximations.

Taking into account that the associated Legendre polynomials are connected with the Bessel functions of arbitrary order, we can show, as in Ref. 18, that for small sighting angles Eq. (19) is equivalent to the reduction of RTE to the form

$$(\hat{l},\nabla) = (\hat{l},\hat{z})\frac{\partial}{\partial\tau} + (l_\perp,\nabla_\perp) \approx \left[\mu_0 + (l_\perp, l_{0\perp})\right]\frac{\partial}{\partial\tau}, \qquad (24)$$

where $\nabla_\perp$, $l_\perp$ are the gradient and projection of $\hat{l}$ in the direction, perpendicular to $\hat{l}_0$.

If we use the third form of the SAA, then after a Fourier transform with the allowance made for the ratio between the expansion in terms of spherical functions and the Fourier transform[18] we obtain the equation equivalent to Eq. (19). However, Eq. (24) shows that the solution (23) obtained should allow the description of the rotation of the brightness body from the direction of incidence in the near-surface layer to the vertical direction in the medium depth. If we examine Eq. (4), defining the physical content of the small angle approximation, then we can see that this rotation of the brightness body is taken into account in MSH in calculating the convolution $\Phi_n(\hat{l})$. The neglect of the path variance of the scattered rays remains, as follows from Eq. (23), and it is easy to illustrate analytically with the spatial irradiance in the case of the Henyey–Greenstein scattering phase function as an example:

$$E_0(\tau) = \oint L(\tau,\mu,\varphi)\,d\hat{l} = C_0^0(\tau) =$$

$$= \exp\left[-\frac{\tau}{\mu_0}\left(1 - \frac{\Lambda}{1 - a\ln g}\right)\right]. \qquad (25)$$

It can easily be seen that, as compared with Ref. 8, MSH leads to a decrease in the effective Λ in the medium upon the increase of the angle of incidence or the effective path length of light absorption.

To determine the accuracy and the applicability domain, we have compared the light fields calculated by Eq. (23) and by the numerical method from Ref. 3 with the widely varying optical and geometrical parameters of the medium.

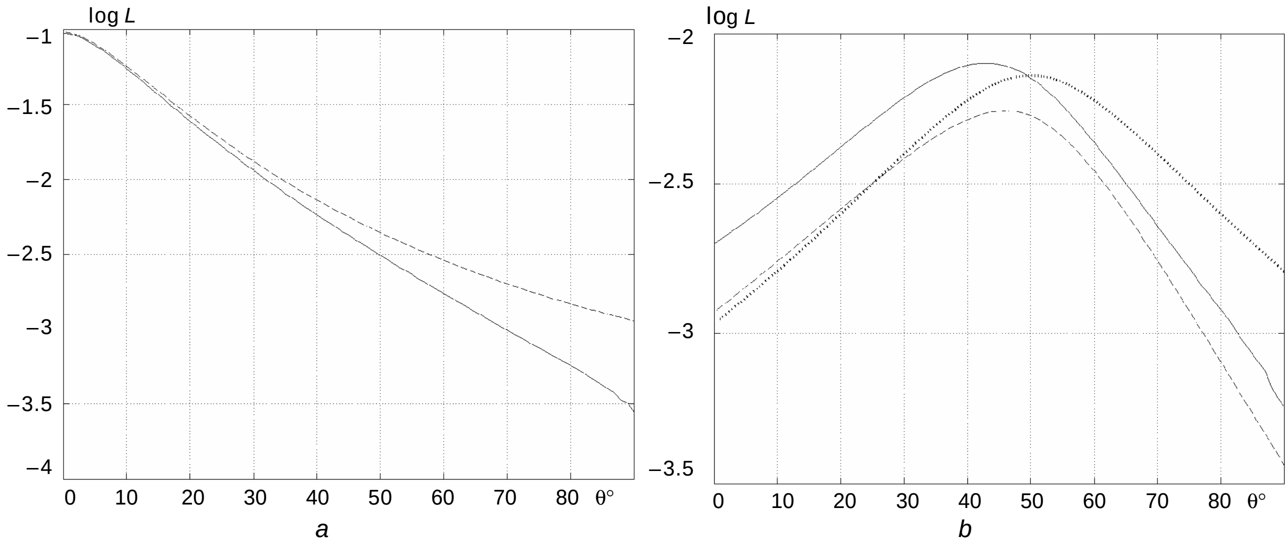

Fig. 1. Brightness body of the light field in a turbid medium (Λ = 0.8, g = 0.97, τ = 15): (*a*) θ_0 = 0°, (*b*) θ_0 = 50°.

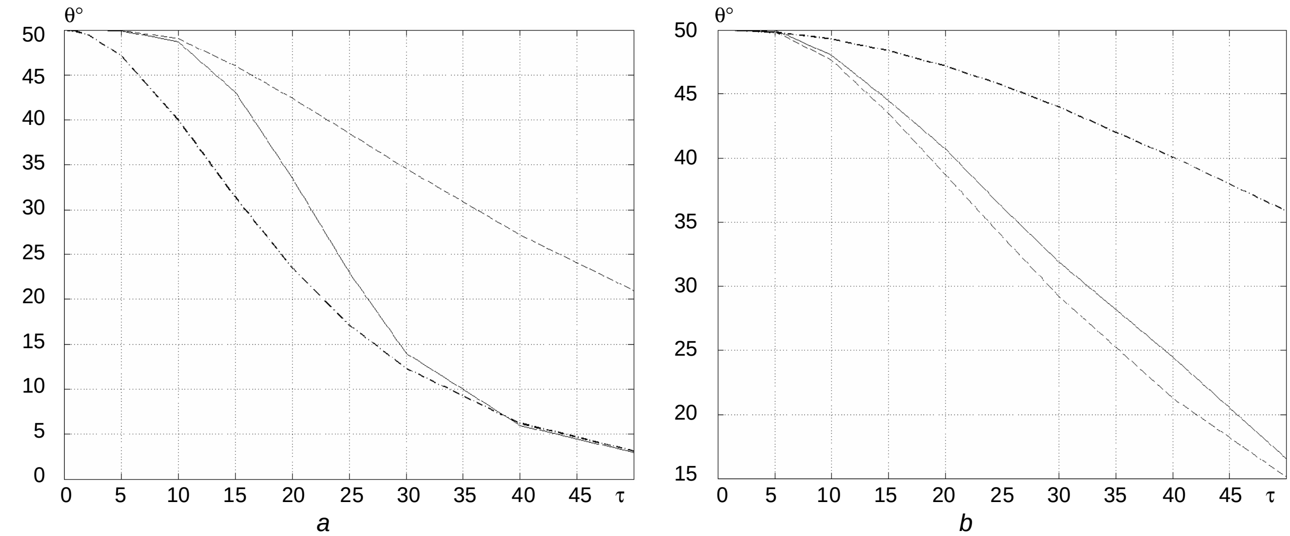

Fig. 2. Rotation of the maximum of the brightness body in a turbid medium (g = 0.97, θ_0 = 50°) with the depth: Λ = 0.8 (*a*) and 0.99 (*b*).

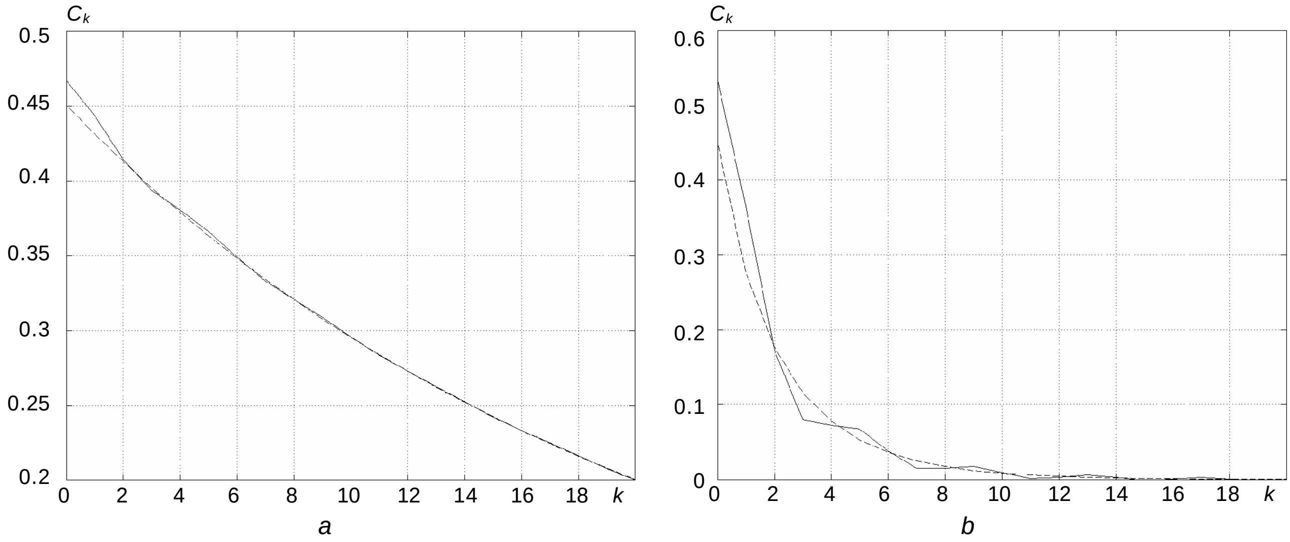

Fig. 3. Amplitudes of harmonics of the angular spectrum of the brightness body in a turbid medium (Λ = 0.8, θ_0 = 50°) as functions of the number: g = 0.97, τ = 10 (*à*); g = 0.7, τ = 1 (*b*).

Figures 1 to 3 compare the light fields calculated in a turbid medium in the small angle approximation with the exact numerical calculation by the method from Ref. 3. The exact solution is shown by the solid line, the solution by the MSH is shown by the dashed line, and SAA is shown by dots, while the dot-and-dash line shows the modification of the diffusion approximation with partial account for the variance of ray trajectories.[1] Analysis of the dependences shown allows the following conclusions to be drawn:

1. The applicability domain of the small angle approximation is the whole forward hemisphere of sighting angles, and therefore the term "small-angle" reflects the idea of the assumption, forming the basis for this approximation, rather than its applicability domain.

2. The change of the optical characteristics of the medium (increase of Λ or decrease of the scattering anisotropy) narrows the applicability domain, but it rather well describes the solution in the range of either smaller angles or smaller optical depths. The range of validity of the MSH is determined by the requirement of anisotropy of the brightness body, when the main contributors are higher harmonics, rather than the scattering anisotropy.

3. Consequently, the MSH has an intermediately asymptotic, rather than asymptotic character, when the approximate solution approximated the exact one virtually everywhere in its range of definition.

4. The diffusion approximation with the allowance for the variance of the ray trajectories[1] rather well describes the integral parameters of the field, while being unsuitable for the description of the brightness body (converges on average, but does not in uniform metrics).

5. The MSH is the most general form of the small angle approximation, all other forms follow from it in the case of small sighting angles or strong scattering anisotropy.

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