CUMULANT METHOD FOR ESTIMATING THE EFFICIENCY OF A SEGMENTED ADAPTIVE MIRROR

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The operating efficiency of an adaptive optics system depends strongly on the quality of a wavefront approximation by an adaptive mirror. This paper is devoted to the development of a new method of analyzing the operating efficiency of a segmented adaptive mirror. In the method, the cumulant efficiency criterion is used with Poisson noise in control channels.

1. INTRODUCTION

At the present time segmented adaptive mirrors are widely used[1,2] in the design of adaptive optics systems intended for compensation for nonstationary aberrations of laser beams propagating in the turbulent atmosphere. This is primarily due to high quality of correction comparable with that of a flexible mirror in spite of a step-wise approximation of a wavefront. In the design of the adaptive optics systems on the basis of such mirrors the problem arises of evaluation of their efficiency. Assessment of potential characteristics of an adaptive mirror, when compensating for phase distortions of the optical radiation wavefront, allows one to predict the efficiency of a system with such a mirror. Considering this problem, one should subdivide the possible errors into two types. The first type of error occurs due to finite number of actuators of a segmented adaptive mirror. The technique of evaluation of such errors is described in Ref. 3. Let us call them the errors of the approximation. The errors of the second type are due to noise in control channels of adaptive optics system. Further we refer to them as noisy errors. The peculiarity of these two types of errors is the following. The larger is the number of adaptive optics system actuators, the lower is the value of the first error, and the higher is the value of the second error. So the typical problem of optimization arises in the form

$$\min_{N} \sigma^2 = \sigma_{\rm a}^2 + \sigma_{\rm n}^2, \qquad (1)$$

where σ^2 is the standard deviation of the total error, $\sigma_{\rm a}^2$ is the standard deviation of the approximation error, and $\sigma_{\rm n}^2$ is the standard deviation of the noisy error.

It should be noted that from the point of view of an analysis of the efficiency, most interesting is the case when Poisson noise arising from shot noise of photodetectors is present in the control channels of the adaptive optics system. It is precisely this situation that arises in an analysis of the efficiency of multidither system with a segmented mirror.

In this paper, a radically new approach has been proposed to evaluate the operating efficiency of a segmented mirror used in the adaptive optics system. The approach is based on the cumulant criterion of efficiency.

The first advantage of the cumulant analysis technique[4] is clearly defined statistical meaning of cumulants, in contrast to moments of random variables. They are uncorrelated and can be defined individually. This leads to the simple dependence of various mean values at the output of nonlinear system on the cumulants of the input parameters. The second advantage of cumulants is connected with the fact that allowance for cumulants of higher orders allows one to describe easily any non-Gaussian random variable. It is just this feature that makes the cumulant technique so valuable for description of non-Gaussian variables such as Poisson ones. It should be noted that a real function that approximates the probability distribution always corresponds to a finite set of cumulants, whereas a non-singular function with zero higher-order moments is nonexistent. This fact becomes very important for approximate representation of random variables that have only a finite set of cumulants. Precisely this approach is suitable for investigation of the transforms of Poisson random variables that characterize processes in the adaptive optics systems.

2. CUMULANT EFFICIENCY CRITERION

Let us consider the problem of analyzing the efficiency of a segmented mirror in the following statement. Let us introduce the Euclidean space P with the scalar product of the form

$$\int U(r)\; V(r)\; {\rm d}r = (U,\; V) \qquad (2)$$

with the norm

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$$||U, V|| = (U, V)^{0.5}. \quad (3)$$

Let us consider a system of orthonormal functions S such as

$$(S_i, S_j) = 0 \quad \text{for } i \neq j,$$
$$(S_i, S_j) = 1 \quad \text{for } i = j. \quad (4)$$

in space P.

The surface of an adaptive mirror can be represented as a linear combination of functions[3] $S_i(r)$ further referred to as response functions of the adaptive mirror

$$W(r) = \sum_{i=1}^{N} S_i(r)\, a_i, \quad r \in \Omega, \quad (5)$$

where $W(r)$ is the function that describes the profile of the segmented adaptive mirror, a_i is the control signal in the ith control channel, Ω is the aperture plane of the adaptive optics system, and N is the number of actuators.

Let us write the response functions $S_i(r)$ in the following form:

$$S_i(r) = 1 \quad \text{for } r \in \Omega_i,$$
$$S_i(r) = 0 \quad \text{for } r \notin \Omega_i, \quad (6)$$

where Ω_i is the surface of the ith subaperture of the adaptive optics system.

Principally, the subaperture of the adaptive optics system can have arbitrary shape, for example, hexagonal, circular, or rectangular. For the last case the response function, for example, has the form

$$S_i(r) = 1 \quad \text{for } x \in [x_i + \Delta x] \;\; y \in [y_i + \Delta y], \;\; r = \{x, y\};$$
$$S_i(r) = 0 \quad \text{for } x \notin [x_i + \Delta x] \;\; y \notin [y_i + \Delta y]. \quad (7)$$

Subscripts i and j are commonly used to describe subapertures of different shapes. However, to reduce the number of subscripts for the finite number of subapertures they always can be re-numerated. It also should be noted that $W(r)$ differs from the sought-after phase profile $\Phi(r)$. This difference referred to as the error of the approximation is not considered in the present paper.

The limits set on the dynamic range of control signals can be ignored since the range of variation of displacements of piezoceramic actuators of modern mirrors always satisfies the requirements (as a rule, they are $\pm 2\lambda_1$, where λ_1 is a wavelength).

Irrespective of a segmented mirror control algorithm, the additive noise is always present in the control channels of the adaptive system. This noise will be described by a set of random variables n_i. It should be noted that mathematical calculations can be essentially simplified in such approach, and the results obtained can be generalized on stationary random processes. Let us assume that a set of cumulants κ describing a set of Poisson random variables n_i is known. All combined cumulants κ_{ij} should be taken zero in this case, and other cumulants - equal to λ, where λ is the parameter of Poisson distribution. For real adaptive system the cumulant characteristics always can be recalculated to the input of the segmented mirror. Then the noisy error can be written as follows:

$$\Delta W(r) = W(r) - W^*(r, n), \quad r \in \Omega$$

or

$$\Delta W(r) = \sum_{i=1}^{N} S_i(r)\, a_i - \sum_{i=1}^{N} S_i(r)\, (a_i + n_i) = \sum_{i=1}^{N} S_i(r)\, n_i, \quad (8)$$

where $W^*(r, n)$ is the mirror response with noisy control channels.

After averaging of the error over the aperture of the adaptive optics system, we obtain

$$\Delta W_s(r) = 1/s \int_{\Omega} \sum_{i=1}^{N} S_i(r)\, n_i \,\mathrm{d}r. \quad (9)$$

Apparently $\Delta W_s(r)$ is the random variable. Earlier in Refs. 3 and 5 it was *a priori* assumed that its mathematical expectation is equal to zero

$$M[(\Delta W_s(r))] = m_W = 0. \quad (10)$$

In this case, only the second central moment (variance) $M[(\Delta W_s(r) - m_W)^2]$ was considered. However, the relation $m_W = 0$ is not fulfilled for real adaptive optics systems, and taking into account the fact that the higher moments of random variable n_i are non-Gaussian, the question about the values of the higher moments of random variable $\Delta W_s(r)$ is still an open question. In such a situation it is expedient to consider the efficiency criterion of the segmented mirror employed in adaptive optics system

$$\theta_k = \left\langle \left(1/s \int_{\Omega} \sum_{i=1}^{N} S_i(r)\, n_i \,\mathrm{d}r\right)^k \right\rangle, \quad k = 1, L, \quad (11)$$

where the angular brackets denote averaging over an ensemble of realizations.

In essence, the components of vector $\boldsymbol{\theta}$ are the higher-order moments of random variable $\Delta W_s(r)$. In what follows we obtain and analyze the relations of the form

$$\theta_k = f\,(S_i(r)\, n_i\,, N, \kappa)\,.$$

As it occurred, such description is very convenient. It allows us to obtain the analytical equations for many practically important cases.

3. DERIVATION OF BASIC RELATIONS

At $k = 1$, Eq. (11) takes the form

$$\theta_1 = <(1/s \int_\Omega \sum_{i=1}^{N} S_i(r)\ n_i\ dr)> = N\ \alpha_1, \quad (12)$$

where α_1 is the first-order moment of the random variable n_i, s is the area of the adaptive system aperture.

In the presence of Poisson noise in the control channels of adaptive optics system we have $\alpha_1 = \lambda = \kappa_1$, where λ is the parameter of Poisson distribution.

We can write the following equation for θ_2

$$\theta_2 = <(1/s \int_\Omega \sum_{i=1}^{N} S_i(r)\, n_i\, dr_1 \times 1/s \int_\Omega \sum_{j=1}^{N} S_j(r)\, n_j\, dr_2)> . \quad (13)$$

In accordance with the properties of cumulants of Poisson random variables, in Eq. (13) we have

$$<n_i\ n_i> = \alpha_2 = \lambda^2 + \lambda, \quad (14)$$

where α_2 is the second-order moment of a random variable n_i.

Taking into account Eq. (14) and the fact that according to Eq. (4) for $i \neq j$ all products $S_i S_j$ are equal to zero, Eq. (13) is written as

$$\theta_2 = N\ (\lambda^2 + \lambda). \quad (15)$$

For θ_3 we derive the following expression:

$$\theta_3 = <(\sum_{i=1}^{N} \sum_{j=1}^{N} \sum_{k=1}^{N} \int_\Omega \int_\Omega \int_\Omega S_m\ S_n\ S_r\ dr_1\ dr_2\ dr_3\ n_i\ n_j\ n_k)>. \quad (16)$$

According to the properties of cumulants of Poisson random variables, the product $<n_i\, n_i\, n_i>$ in Eq. (16) is

$$<n_i\ n_i\ n_i> = \alpha_3 = \kappa_3 + 3\ \kappa_1\ \kappa_2 + \kappa_1^3 = \lambda^3 + 3\ \lambda^2 + \lambda. \quad (17)$$

Taking into account Eqs. (4) and (17), after analogous manipulations we have

$$\theta_3 = N\ (\lambda^3 + 3\ \lambda^2 + \lambda). \quad (18)$$

Expression for θ_4 is written as follows:

$$\theta_4 = <(\sum_{i=1}^{N}\sum_{j=1}^{N}\sum_{k=1}^{N}\sum_{l=1}^{N} \int_\Omega \int_\Omega \int_\Omega \int_\Omega {}_m S_n S_r S_t dr_1 dr_2 dr_3 dr_4 n_i n_j n_k n_l)>. \quad (19)$$

Then for Poisson random variables we have

$$<n_i\ n_i\ n_j\ n_j> = \alpha_2^2 = \lambda^4 + 2\ \lambda^3 + \lambda^2. \quad (20)$$

Taking into account Eq. (20), Eq. (19) can be written as

$$\theta_4 = N\ (\lambda^4 + 2\ \lambda^3 + \lambda^2). \quad (21)$$

Analogously, considering the properties of orthogonality for step-wise response functions and using the corresponding equations for the higher-order moments (expressed in terms of the cumulants of Poisson distribution) of random variable n_i, it is possible to obtain the corresponding analytical expressions for θ_k at the given k.

4. PROBABILITY DENSITY OF NOISY ERROR DISTRIBUTION FOR A HYPOTHETICAL MIRROR

Taking into account normalization by N, let us approximate the components of vector θ by the expression

$$\theta_k = \sum_{m=1}^{k} (m\ \lambda)^{k-m} + \lambda^k. \quad (22)$$

Such an approximation is close enough to Eqs. (12), (15), (18), and (21) especially for $\lambda \gg 1$. The characteristic function of this approximation can be written as follows[4]:

$$Q(i\ u) = 1 + \sum_{k=1}^{\infty} \theta_k (i\ u)^k / k!. \quad (23)$$

Upon substituting Eq. (22) into Eq. (23), we obtain

$$Q(i\,u) = 1 + \sum_{k=1}^{\infty} \lambda^k (i\,u)^k / k! + \sum_{k=1}^{\infty} \sum_{m=1}^{k} (m\,\lambda)^{k-m} (i\,u)^k / k!. \quad (24)$$

The first two terms of Eq. (24) can be written in the following form:

$$1 + \sum_{k=1}^{\infty} \lambda^{2k} (-\ 1)^k\ (u)^{2k} / (2k)! +$$

$$+ \sum_{k=1}^{\infty} \lambda^{2k-1} (-\ 1)^k\ (u)^{\ 2k-1} / (2k-1)! =$$

$$= \cos(\lambda\ u) + i\ \sin(\lambda\ u) = \exp\ (i\ \lambda\ u). \quad (25)$$

Let us transform the third term of Eq. (24) in the following way:

$$\sum_{k=1}^{\infty}\sum_{m=1}^{k}(m\,\lambda)^{k-m}\,(i\,u)^{k}/k! =$$

$$= \sum_{m=1}^{k}(m\,\lambda)^{-m}\,(\sum_{k=1}^{\infty}(m\,\lambda)^{2k}\,(-1)^{k}\,(u)^{2k}/(2k)! +$$

$$+ i\sum_{k=1}^{\infty}(m\,\lambda)^{2k-1}\,(-1)^{k}\,(u)^{2k-1}/(2k-1)!) =$$

$$= \sum_{m=1}^{k}(\exp\,(i\,m\,\lambda\,u) - 1)/(\lambda\,m)^{m},\quad k\to\infty. \qquad (26)$$

Finally, the characteristic function of the sought-for probability density is

$$Q(i\,u) = \exp\,(i\,\lambda\,u) + \sum_{k=1}^{\infty}(\exp\,(i\,m\,\lambda\,u) - 1)/(\lambda\,m)^{m}. \qquad (27)$$

To find out the non-normalized probability density of distribution P^* we must calculate the reverse Fourier transform of the characteristic function $Q(iu)$

$$P^*(x) = 1/2\,\pi\int_{-\infty}^{\infty}Q(i\,u)\,\exp\,(i\,x\,u)\,\mathrm{d}x. \qquad (28)$$

Upon substituting Eq. (27) into Eq. (28), we obtain

$$P^*(x) = 1/2\,\pi\int_{-\infty}^{\infty}(\exp\,(i\,\lambda\,u) +$$

$$+ \sum_{m=1}^{\infty}(\exp\,(i\,m\,\lambda\,u) - 1)/(\lambda\,m)^{m})\,\exp\,(i\,x\,u)\,\mathrm{d}x =$$

$$= \delta(\lambda - x) + \sum_{m=1}^{n}(\delta(m\,\lambda - x) - \delta(x))/(\lambda\,m)^{m},\ n = \overline{1,\infty}\,. \qquad (29)$$

The final expression for the probability density P^* is

$$P^*(m) = 1/(\lambda\,m)^{m}. \qquad (30)$$

To calculate the normalization factor of the sought-for distribution density, let us sum up series (30) as $m\to\infty$

$$\sum_{m=1}^{\infty}1/(\lambda\,m)^{m} = 1/m!\,\lambda^{m} = \exp\,(1/\lambda) - 1. \qquad (31)$$

On account of normalization condition we obtain the following expression for the distribution density:

$$P(m) = 1/((\lambda\,m)^{m}\,\exp\,(1/\lambda - 1)). \qquad (32)$$

Thus, as could be expected, the obtained distribution density is discrete. The form of the probability density for different λ is shown in Fig. 1.

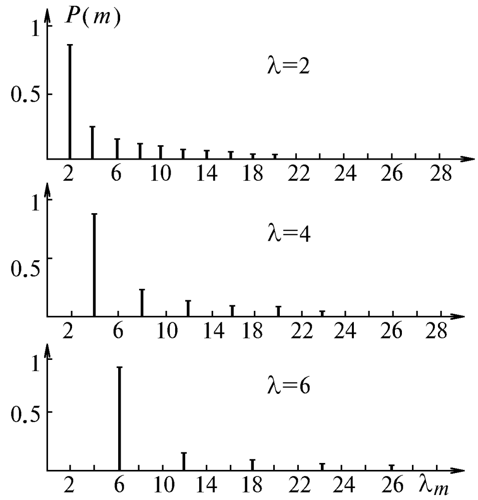

FIG. 1.

5. CONCLUSIONS

Choosing the number of control channels with allowance for the quality of the wavefront approximation by a segmented mirror, at the stage of design of the adaptive optics system it is expedient to use the developed technique for evaluating the efficiency of a flexible adaptive mirror. To obtain the exact values of cumulant criterion elements in the calculation of the sums given by Eqs. (12), (15), (18), and (21), it is rational to use the specific values of the higher-order moments calculated with the use of the set of cumulants of corresponding random variables (for example, Poisson random variables). It should be specially emphasized that the developed method is general. For example, using the actual parameters of noise distribution in the control channels, it would be easy to derive the corresponding analytical expressions for the components of vector **θ** for the set of cumulants of non-Poisson distribution.

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