Thermal radiation of weakly connected oscillator system subjected to permanent stochastic disturbance

V.N. Ivanov and I.V. Ivanov

Omsk State Technical University

Received October 24, 2005

The feedback effect of oscillators, interacting with the environment, on population of their quantum states and on thermal radiation is considered. Self-acting of the oscillators is simulated by the nonlinear operator in the Schrödinger equation. The oscillator stationary eigenstates are shown to be the most probable at permanent stochastic disturbance. Moreover, the change of quantum states can occur only as a jump. The Bose condensation of quantum states takes place in oscillators at $T < \hbar\omega/(2k)$ (ω is the oscillation frequency; k is the Boltzmann constant). The numerical modeling shows that at a higher T the probability population distribution for oscillators interacting with the environment is rather close to the Boltzmann distribution. At relatively low T, radiation is caused mainly by quantum transitions between the nearest levels. When temperature increases, the radiation energy maximum of oscillators is shifted to the short-wave part of the spectrum.

Introduction

A quantum oscillator is the well-investigated object used for description of vibrations in physical systems. In actual systems, oscillators usually represent parts of complex formations such as clusters or Van der Waals molecules with a large collision cross section, which leads to their permanent stochastic disturbance. The same stochastic disturbance experiences any quantum system due to interaction with vacuum state of the electromagnetic field. Therefore, when describing the quantum oscillator states, it is necessary to take into account this permanent disturbance. This can be done when going from the Neumann equation for the statistical operator of a complex system to the equation for the density matrix of a single oscillator.

However, such reduction causes a necessity in prior representation of the collision integral owing to any model of the oscillator interaction with the environment. In a number of cases, when writing down the kinetic equations, it can lead to distortion of the information about the physical system. Besides, such approach has limited analytical opportunities due to necessity to take into account a relatively large number of the differential equations proportional to the squared number of energy levels. In this connection, it is desirable to construct such a scheme for description of disturbed quantum oscillator states, which would conserve the computational capabilities of the Schrödinger equation. Nevertheless, the wave functions should be determined so that the mean values of physical quantities were close to the values obtained by means of the density matrix formalism.

One of the possible approaches[1–3] to solve this problem is considered in this paper, where a separated subsystem state interacting with the environment is described by the effective wave function averaged by the environmental effect. Such effective functions are the result of solution of the nonlinear Schrödinger equation.

Schrödinger equation for quantum subsystem subjected to permanent stochastic disturbance

Constructing the Schrödinger equation for effective wave functions is carried out by the Feynman method,[4] which is one of the ways for description of wave propagation in the medium. The Feynman method considers the ψ-function (wave surface) as a source of secondary waves. Each secondary wave $K(\mathbf{r}, \mathbf{r}_1)$ (propagator) represents the sum connecting the start and terminal points by all possible ways and is written down as a functional integral

$$K(\mathbf{r}, \mathbf{r}_1) = \int_{-\infty}^{\infty} A \exp\left(\frac{i}{\hbar} S(\mathbf{r}, \mathbf{r}_1, t+\tau, t)\right) D\mathbf{r}(t). \qquad (1)$$

In Eq. (1), $S(\mathbf{r}, \mathbf{r}_1, t+\tau, t)$ is the classical action of calculation along the path connecting the points $\mathbf{r}_1$ and $\mathbf{r}$; t is the zero time, and $t+\tau$ is the finite time; A is the amplitude. At such assignment of the propagator, it fits the law of the group multiplication:

$$K(\mathbf{r}, \mathbf{r}_1) = \int_{-\infty}^{\infty} K(\mathbf{r}, \mathbf{r}_2) K(\mathbf{r}_2, \mathbf{r}_1) \mathrm{d}^3\mathbf{r}_2, \qquad (2)$$

which allows writing down the integral equation equivalent to the Schrödinger equation for the wave function of a separated subsystem[4]:

0235-6880/07/01 24-08 $02.00

$$\psi(\mathbf{r},t+\tau)=\int\limits_{-\infty}^{\infty}A\exp\left(\frac{i}{\hbar}S(\mathbf{r},\mathbf{r}_1,t+\tau,t)\right)\psi(\mathbf{r}_1,t)D\mathbf{r}(t)\mathrm{d}^3\mathbf{r}_1. \tag{3}$$

In the Feynman method, each possible trajectory is a random polygonal line, whose realization is caused by quantum fluctuations. The functional integral from Eq. (3) allows one to include the permanent stochastic disturbance of the separated subsystem into reasons affecting the form of the path. Actually, if to consider every alternative path starting at $\mathbf{r}_1$ and finishing at $\mathbf{r}$ as a result of Brownian motion, the disturbing factors can be accounted for by the function

$$W\left(\mathbf{r},\mathbf{r}_1,\mathbf{U},\frac{\tau}{2}\right)=C\exp\left(-\frac{(\mathbf{r}-\mathbf{r}_1)^2}{2p\beta^{-2}\tau}+\frac{\left((\mathbf{r}-\mathbf{r}_1)\cdot\mathbf{U}_0\right)}{2p\beta^{-2}}-\frac{\mathbf{U}_0^2\tau}{8p\beta^{-2}}\right)\times$$

$$\times\exp\left(-\frac{\mathbf{U}^2}{2p\beta^{-1}}+\frac{\left[\mathbf{U}\cdot\left(\mathbf{r}-\mathbf{r}_1-\mathbf{U}_0\frac{\tau}{2}\right)\right]}{p\beta^{-1}\tau}\right), \tag{4}$$

being the solution of the Fokker–Planck equation.[5] This function finds a joint probability distribution for the time $\tau/2$ with one or another velocity at the given point of the space (it is assumed that this time is much less than typical times of change of the time-regular quantum subsystem disturbance). In Eq. (4), C is the normalizing constant; $p = \beta kT^*/m$, where m is the mass of the particle participating in Brownian motion; k is the Boltzmann constant, T^* is the effective temperature of the environment. If collisions with the environment is the main factor determining the form of trajectories, hence, T^* corresponds to the standard temperature. The parameter β is the viscosity of the environment. Henceforth, assume that β changes only adiabatically. Physically, the vector $\mathbf{U}_0$ is the ordered motion velocity of the Brownian particle in the environment: in case of normal diffusion, $\mathbf{U}_0\frac{\tau}{2}$ corresponds to the central moment of the Gaussian distribution function.

The distribution (4) allows determining up to a factor the contribution to the final ψ-function of paths starting at the moment t at the point $\mathbf{r}_1$ and finishing at $t + \tau$ at the point $\mathbf{r}$:

$$K(\mathbf{r},\mathbf{r}_1)=A\int\limits_{-\infty}^{\infty}\exp\left[\frac{i}{\hbar}L\left(\mathbf{r}_1+\mathbf{U}\frac{\tau}{2},\mathbf{U},t+\frac{\tau}{2}\right)\tau\right]\times$$

$$\times W\left(\mathbf{r},\mathbf{r}_1,\mathbf{U},\frac{\tau}{2}\right)\mathrm{d}^3\mathbf{U}. \tag{5}$$

In Eq. (5), according to the standard algorithm,[4] the action was changed by the product of the average Lagrangian $L(\mathbf{r},\mathbf{U},t)$ at the time interval τ by the time interval. Representation of the propagator in the form of Eq. (5) and averaging $A\exp\left[\frac{i}{\hbar}L\left(\mathbf{r}_1+\mathbf{U}\frac{\tau}{2},\mathbf{U},t+\frac{\tau}{2}\right)\tau\right]$ over all possible velocity values are, as a matter of fact, equivalent to the functional integration in Eq. (3). Moreover, if the integration over velocity does not disturb the group properties of Eq. (2), then Eq. (3) automatically is transformed into the integral equation for the effective wave functions averaged by the environmental effect of the separated subsystem:

$$\psi(\mathbf{r},t+\tau)=\int\limits_{-\infty}^{\infty}K(\mathbf{r},\mathbf{r}_1)\psi(\mathbf{r}_1,t)\mathrm{d}^3\mathbf{r}_1. \tag{6}$$

The particular form of the propagator in Eq. (6) is determined by the Lagrangian form. In the considered case, the Lagrangian has two peculiarities. First, an apparent form of the stochastic term cannot be pointed out since it disappears in the finite expression for the action. Actually, if τ is much higher than the duration of the stochastic disturbance fluctuations, then the following equation is valid for the action

$$S(\mathbf{r},\mathbf{r}_1)=\int\limits_{t}^{t+\tau}L(\mathbf{r},\mathbf{U},t_1)\mathrm{d}t_1=\int\limits_{t}^{t+\tau}\left(\tilde{T}-V(t_1)-\tilde{V}(t_1)\right)\mathrm{d}t_1=$$

$$=\int\limits_{t}^{t+\tau}\left(\tilde{T}-V(t_1)\right)\mathrm{d}t_1,$$

where $\tilde{T}$ is the kinetic energy and $V(t) + \tilde{V}(t)$ is the potential energy of the subsystem; $V(t)$ includes all the time-regular terms; $\tilde{V}(t)$ is the stochastic disturbance. Second, except for the standard terms, the Lagrangian should include the expression providing the normalization conservation of the effective wave function in time. Taking into account that the Lagrangian is determined up to a total derivative of any function of coordinate and time,[6] assume that, in a general case, this term can be the functional of $\Phi(\psi)$.

Then

$$L\left(\mathbf{r}_1+\mathbf{U}\frac{\tau}{2},\mathbf{U},t+\frac{\tau}{2}\right)=\frac{m\mathbf{U}^2}{2}-q\varphi\left(\mathbf{r}_1+\frac{\mathbf{U}\tau}{2},t+\frac{\tau}{2}\right)+$$

$$+\frac{q}{c}\left[\mathbf{U}\cdot\mathbf{A}\left(\mathbf{r}_1+\frac{\mathbf{U}\tau}{2},t+\frac{\tau}{2}\right)\right]-\Phi(\psi), \tag{7}$$

where m and q are the mass and the charge characterizing the subsystem; c is the light velocity, $\varphi(\mathbf{r},t)$ and $\mathbf{A}(\mathbf{r},t)$ are the scalar and vector potentials of the electromagnetic field.

Substitution of Eq. (7) into Eq. (5) leads to the following expression for the propagator:

$$K(\mathbf{r},\mathbf{r}_1) = B_1 \exp\left\{ i\frac{m(\mathbf{r}-\mathbf{r}_1)^2}{2\hbar\tau}(1+i\alpha) + \right.$$
$$+i\frac{q}{\hbar c}\left[(\mathbf{r}+\mathbf{r}_1)\cdot\left(\mathbf{A}\left(\frac{\mathbf{r}+\mathbf{r}_1}{2},t+\frac{\tau}{2}\right)-\frac{mc}{2q}(1+i\alpha)\mathbf{U}_0\right)\right]-$$
$$-i\frac{q}{\hbar}\left[\varphi\left(\frac{\mathbf{r}+\mathbf{r}_1}{2},t+\frac{\tau}{2}\right)+\frac{1}{2c}\left(\mathbf{U}_0\cdot\mathbf{A}\left(\frac{\mathbf{r}+\mathbf{r}_1}{2},t+\frac{\tau}{2}\right)\right)-\right.$$
$$\left.\left.-\frac{m}{8}(1+i\alpha)\mathbf{U}_0^2\right]\tau-\chi\frac{m}{\hbar^2}(\mathbf{r}-\mathbf{r}_1)^2-i\frac{\tau}{\hbar}\Phi(\psi)\right\}, \qquad (8)$$

where B_1 is the constant; $\alpha = \hbar\beta/(kT^*)$, $\chi = kT^*/2$.

The propagator (8) satisfies the condition (2) and, therefore, can be used for determination of the effective wave functions.

The transition from the integral equation (6) to the Schrödinger equation is carried out owing to the Feynman standard algorithm. The right part of Eq. (6) [propagator is determined by the formula (8)] is interpreted as an integro-differential operator acting on $\psi(\mathbf{r}, t+\tau)$. It is assumed that this operator is unity accurate to $O(\tau^2)$. It is possible only at its action on functions satisfying the differential Schrödinger equation.[4] In this case, the operator shows itself as unit when acting on functions satisfying the equation:

$$i\hbar\frac{\partial\psi}{\partial t} = \frac{1}{1+i\alpha}\left[\frac{1}{2m}\left(\hat{\mathbf{P}}-\frac{q}{c}\mathbf{A}-(1-i\alpha)\frac{m\tilde{\mathbf{U}}_0}{2}\right)^2+\chi\right]\psi+$$
$$+q\left(\varphi-\frac{(\mathbf{A}\cdot\tilde{\mathbf{U}}_0)}{2c}-(1+i\alpha)\frac{m\tilde{\mathbf{U}}_0^2}{8q}\right)\psi+\Phi(\psi)\psi, \qquad (9)$$

where $\hat{\mathbf{P}}$ is the momentum operator.

The Schrödinger equation written down for the effective wave functions, in a general case, is nonlinear due to the functional $\Phi(\psi)$. The quantity $\tilde{\mathbf{U}}_0 = -\mathbf{U}_0$ in Eq. (9) determines the movement velocity of the environment relative to the separated quantum subsystem, in the considered case this velocity is zero. By formation, ψ-functions satisfying the obtained Schrödinger equation correspond to the subsystem states averaged by the environmental effect. It is shown in Ref. 7 that, at $\Phi(\psi) = 0$ and $\tilde{\mathbf{U}}_0 = 0$, the same in form Neumann equation follows from Eq. (9) for the statistical operator $\hat{\rho} = |\psi\rangle\langle\psi|$, as for the subsystem density matrix separated by the Lacks method[8] at a density matrix reduction of a complex system, in which the second subsystem behaves like a Markovian thermostat.

The standard unitary transformation[9] leads the Eq. (9) to the form valid in the dipole approximation:

$$i\hbar\frac{\partial\psi}{\partial t} = \frac{1}{1+i\alpha}\left(\frac{1}{2m}\hat{\mathbf{P}}^2+\chi\right)\psi+q\varphi\psi+\Phi(\psi)\psi-(\mathbf{E}\cdot\mathbf{d})\psi, \qquad (10)$$

where $\mathbf{E}$ is the intensity of the external electric field; $\mathbf{d}$ is the dipole moment of the quantum subsystem.

In order to write down an apparent form of the functional $\Phi(\psi)$, we separate an effective operator of the subsystem disturbance by the environment in Eq. (10) (the term in the Hamiltonian disappearing at $\alpha \to 0$):

$$\hat{V} = -\frac{i\alpha}{1+\alpha^2}\left(\frac{1}{2m}\hat{\mathbf{P}}^2+\chi\right)+\Phi(\psi). \qquad (11)$$

Since oscillators are stable formations, their wave functions should conserve the normalization in time. Therefore, the operator (11) should be Hermitian.[10] Then matrix elements of the functional $\Phi(\psi)$ satisfy the equality

$$\langle\psi|\Phi(\psi)|\psi\rangle-\langle\psi|\Phi^*(\psi)|\psi\rangle = \frac{2i\alpha}{1+\alpha^2}\langle\psi|\left(\frac{1}{2m}\hat{\mathbf{P}}^2+\chi\right)|\psi\rangle. \qquad (12)$$

The real part of matrix elements $\langle\psi|\Phi(\psi)|\psi\rangle$ is the energy of the subsystem interaction with the environment. Within the limits of the considered model, it is stipulated by the stochastic disturbance and at thermodynamic equilibrium of the subsystem, its value should be close to zero.

The relation (12) is always fulfilled, if the functional $\Phi(\psi)$ has the form

$$\Phi(\psi) = \frac{i\alpha}{1+\alpha^2}\langle\psi|\frac{1}{2m}\hat{\mathbf{P}}^2+\chi|\psi\rangle. \qquad (13)$$

Note that at such form of the functional, a part of solutions $\psi(\mathbf{r}, t)$ of Eq. (10) at $\Phi(\psi) \neq 0$ can be expressed through the wave functions $\tilde{\psi}(\mathbf{r}, t)$, being the solution for the same equation[2] at $\Phi(\psi) = 0$:

$$\psi = \frac{\tilde{\psi}}{\sqrt{\langle\tilde{\psi}|\tilde{\psi}\rangle}} = \sum_n C_n(t)\psi_n(\mathbf{r}). \qquad (14)$$

In Eq. (14), $C_n(t)$ are the regular time functions, and $\psi_n(\mathbf{r})$ are the eigenfunctions of the stationary Schrödinger equation:

$$E_n\psi_n = \frac{1}{1+i\alpha}\left(\frac{1}{2m}\hat{\mathbf{P}}^2+\chi\right)\psi_n+q\varphi\psi_n =$$
$$= \frac{1}{1+i\alpha}(\hat{T}+\chi)\psi_n+U\psi_n, \qquad (15)$$

where E_n is the separation constant.

Nonlinearity of Eq. (10) is the natural consequence of reduction of the problem on complex system (separated subsystem and the environment); it is referred to the problem on state of the subsystem only. The functional (13) can be interpreted as a term accounting for the feedback, i.e., self-action of the separated subsystem through the environment. Substitution of the functional (13) into Eq. (9) leads in the Schrödinger equation to the Hamiltonian coinciding in structure with that of the written down in the stationary case when studying the molecules interacting with the environment[11]:

$$\hat{H} = \hat{H}_0 + \lambda_1 \hat{A}\langle\psi|\hat{B}|\psi\rangle,$$

where $\hat{H}_0$ is the Hamiltonian of an isolated molecule; $\hat{A}$ and $\hat{B}$ are the operators, whose choice of the apparent form depends on the accepted model of the molecule interaction with the environment; λ_1 is the parameter characterizing the interaction intensity.

Due to the nonlinearity of Eq. (10), among its solutions are such, whose coefficients $C_n(t)$ in superposition are the irregular time functions[3,8]:

$$\psi(\mathbf{r}, t) = \sum_n C_n(t)\psi_n(\mathbf{r}). \qquad (16)$$

In particular, this fact can take place due to the time-irregular disturbance at α fluctuations. These "additional" solutions determine the population of quantum levels.

In order to make Eq. (16) the solution of nonlinear equation (10) at $\mathbf{E}(\mathbf{r}, t) = 0$, it is sufficient for $C_n(t)$ to satisfy the differential equations:

$$i\hbar\frac{\partial C_n}{\partial t} = E_n C_n + \frac{i\alpha}{1+\alpha^2} C_n \sum_{m,k} C_m^* C_k \tilde{T}_{mk}, \qquad (17)$$

where

$$\tilde{T}_{mk} = \langle\psi_m|\hat{T} + \chi|\psi_k\rangle = T_{mk} + \chi\langle\psi_m|\psi_k\rangle.$$

As follows from Eq. (17), for filling numbers $P_n(t) = |C_n(t)|^2$:

$$\frac{\partial P_n}{\partial t} = \frac{2\alpha}{\hbar(1+\alpha^2)} P_n\left(-\tilde{T}_{nn} + P_n\sum_{m,k}\sqrt{P_m P_k}\left|\tilde{T}_{mk}\right|\cos(\vartheta_{mk})\right). \qquad (18)$$

In Eq. (18), the cosine argument is the phase of the complex number $C_m^*(t)C_k(t)\tilde{T}_{mk}$. If to consider the equilibrium state of the separated quantum subsystem $\left(\frac{\partial P_n}{\partial t} = 0\right)$, then a set of equations derived from Eq. (18) has the solution provided that only one of the numbers P_n (n = 1, 2, 3, ...) is non-zero, i.e., the values of the filling numbers can be only 0 and 1. Therewith, the condition $\sum_n P_n(t) = 1$ holds automatically. This means that the most probable states of the stationary quantum subsystem are those described by the Hamiltonian eigenfunctions of Eq. (15). However, the populations of different quantum levels can vary with time.

When analyzing the variation dynamics of the populated level due to the feedback, it is necessary to follow simultaneously the behavior of all quantum levels including the unpopulated ones. For "pure" states $\psi = C_n(t)\psi_n(\mathbf{r})$, Eq. (17) coincides in form with the well-studied equation[12] describing the one-parameter vector field family in the plane:

$$\frac{\mathrm{d}z}{\mathrm{d}t} = z(i\rho + \varepsilon + kz z^*), \qquad (19)$$

where

$$\varepsilon = -\frac{\alpha}{(1+\alpha^2)\hbar}(T_{nn} + \chi) < 0. \qquad (20)$$

It is known that at $\varepsilon < 0$, point $z = 0$ is a position of the stable equilibrium. It means that if α essentially differs from zero, the unpopulated states of the separated subsystem are stable. Therefore, this subsystem state in the absence of $\mathbf{E}(t)$ can change spontaneously only if ε tends to zero. Moreover, some works[12,13] show that since Eq. (19) has a peculiarity in the form of a fold, when ε approaches zero from the negative side at small but non-zero value of $|\varepsilon| = \delta$, the available disturbances can reject the system out of the vicinity of the equilibrium position. Thus, the system will jump either to another distant equilibrium position, or to any limit cycle or to another more complicated attracting set. The possibility of such "catastrophe"[13] is of great importance. It shows that when changing the environmental density at values of α below a certain critical value, at which the Schrödinger equation (10) still remains nonlinear, a jump-like change in population of quantum levels is possible.

State of oscillators subjected to permanent stochastic disturbance

Dynamics of population in the quantum oscillator levels is determined by a set of parameters in Eq. (15).

It was shown[3] that in the one-dimensional case the oscillator wave functions satisfying Eq. (15) are expressed through the Hermitian polynomials of a complex variable:

$$\psi_n(\xi) = A_n(-1)^n \exp\left(\sqrt{1+i\alpha}\frac{\xi^2}{2}\right)\frac{d^n}{d\xi^n}\exp\left(-\sqrt{1+i\alpha}\xi^2\right)$$

(A_n is the normalizing constant, and $\xi = x\sqrt{m\omega/\hbar}$; x is the usual coordinate), and separation constants are the complex quantities:

$$E_n = \frac{1}{\sqrt{1+i\alpha}}\left(n + \frac{1}{2}\right)\hbar\omega + \frac{\chi}{1+i\alpha} \qquad (21)$$

(ω is the natural vibration frequency of the oscillator, n = 0, 1, 2, 3, ...).

As follows from Eq. (21), for diagonal matrix elements of the kinetic energy, the following equation is valid accurate to $O(\alpha^2)$

$$T_{nn} = \frac{1}{2}\left(n + \frac{1}{2}\right)\hbar\omega.$$

According to Eq. (20), when no time-regular disturbance exists, a change in equilibrium position for the unpopulated oscillator quantum level is possible only at

$$\alpha \le \alpha_n = 2\hbar\delta / \left[\left(n + \frac{1}{2}\right)\hbar\omega + 2\chi\right]. \qquad (22)$$

Therefore, the probability of changing this equilibrium state is proportional to probability for α to fall in the interval between zero and α_n due to the environmental density fluctuations of the oscillator. Since it is assumed that the system is subjected to the permanent stochastic disturbance, and α_n is small, it is possible to write down approximately

$$F(\alpha_n) = \int_0^{\alpha_n} f(\alpha)\mathrm{d}\alpha \approx C_1\alpha_n + O(\alpha^2), \qquad (23)$$

ignoring the exact form of the probability density $f(\alpha)$, assuming only $f(0) = 0$, where C_1 is the constant.

However, the change in population of the oscillator quantum levels at $\alpha \le \alpha_n$ can take place not always. To consider the influence of prerequisites for being out of the equilibrium position on population of energy levels, let us separate two its eigenstates $\psi_n(\mathbf{r}, t)$ and $\psi_m(\mathbf{r}, t)$, the transition between which is possible. Assume for definiteness sake that originally $P_{n0} = 1$; $P_{m0} = 0$. In the two-level approximation, Eq. (18) is a set of equations:

$$\begin{aligned}
\frac{\partial P_n}{\partial t} &= \frac{2\alpha}{\hbar(1+\alpha^2)} P_n\left(-\tilde{T}_{nn} + P_n\tilde{T}_{nn} + P_m\tilde{T}_{mm} + \right.\\
&\left. + 2\sqrt{P_nP_m}\left|\tilde{T}_{nm}\right|\cos(\vartheta_{nm})\right) = F(P_n, P_m),\\
\frac{\partial P_m}{\partial t} &= \frac{2\alpha}{\hbar(1+\alpha^2)} P_m\left(-\tilde{T}_{mm} + P_n\tilde{T}_{nn} + P_m\tilde{T}_{mm} + \right.\\
&\left. + 2\sqrt{P_nP_m}\left|\tilde{T}_{nm}\right|\cos(\vartheta_{nm})\right) = \Phi(P_n, P_m).
\end{aligned} \qquad (24)$$

According to general properties of nonlinear systems on a plane,[14] the stability of the stationary point depends on the right part of the system of equations (24). In particular, the necessary condition for the absence of other equilibrium points nearby is the difference from zero of the Jacobian calculated at $P_{n0} = 1$, $P_{m0} = 0$:

$$\Delta = \begin{vmatrix} F'_{P_n}(P_n, P_m) & F'_{P_m}(P_n, P_m) \\ \Phi'_{P_n}(P_n, P_m) & \Phi'_{P_m}(P_n, P_m) \end{vmatrix}.$$

The second parameter determining the properties of nonlinear system of equations is the superposition calculated under the same conditions

$$\sigma = F'_{P_n}(P_n, P_m) + \Phi'_{P_m}(P_n, P_m).$$

Note that in case that the oscillator is subjected to the permanent stochastic disturbance accurate to small quantities of higher orders, the following holds for these parameters

$$\Delta = (\alpha\omega)^2\left(n + \frac{1}{2} + \frac{\chi}{\hbar\omega}\right)(n - m);$$

$$\sigma = \alpha\omega\left(2n - m + \frac{1}{2} + \frac{2\chi}{\hbar\omega}\right).$$

The attractive sets in the case under consideration can be the closed curves in addition to equilibrium points. However, they appear only under some conditions. In particular, if originally $n > m$, there are no prerequisites for appearance of the closed curves. If the oscillator will be in the superposition state due to the disturbance, then, according to Eqs. (14) and (21), the wave function with time will tend to the wave function of low state conserving the time normalization. It means that in the case of the "catastrophe," the oscillator will simply jump to the lower state. It is naturally to interpret such a jump as a quantum jump.

In case, when the lower energy level is populated, the situation is different. The oscillator output from the equilibrium position at $\Delta < 0$ proceeds along the trajectory in the form of a loop, whose stability depends on a sign of σ (Ref. 14). When σ is positive, the loop is unstable, and in this case, the quantum jump can takes place. If $\sigma < 0$, then, even if the loop passes through another equilibrium point ($P_n = 0$, $P_m = 1$), the oscillator will return in the initial position. Hence, the oscillator can pass only into the upper levels, whose quantum numbers satisfy the condition

$$m \le 2n + 1/2 + 2\chi/(\hbar\omega).$$

This relation points out that when the effective temperature is below the limit

$$T^* < \hbar\omega/(2k) \qquad (25)$$

the Bose-condensation of the oscillator states should be observed.[3]

The distribution of population of different quantum states can be found by numerical modeling. In this case, it is sufficient to determine the ensemble-averaged population probability for each energy level caused by quantum transitions.

According to Eqs. (22) and (23), for the probability of the oscillator transition from the populated n-level to the vacant m-level to accuracy of factor $1/Z_n$, the following equation is valid

$$Q(n,m) = \frac{1}{Z_n}\frac{\eta(n,m)}{\left(1 + \dfrac{\left(m + \dfrac{1}{2}\right)\hbar\omega}{kT^*}\right)}, \qquad (26)$$

where

$$\eta(n,m) = \begin{cases} 1 & \left(m \le 2n + \dfrac{1}{2} + \dfrac{kT^*}{\hbar\omega}\right), \\ 0 & \left(m > 2n + \dfrac{1}{2} + \dfrac{kT^*}{\hbar\omega}\right). \end{cases} \qquad (27)$$

Since the oscillator transition from one level to another is not a determined process, Eqs. (26) and (27) are also valid in case, when the oscillator finally returns to the initial state if there are prerequisites for the state change. Therefore, Z_n is the statistical sum:

$$Z_n = \sum_{m=0}^{\infty} \frac{\eta(m, n) kT^*}{kT^* + \left(m + \frac{1}{2}\right)\hbar\omega}.$$

One of the possible calculation algorithms for the population probability is the following. First, the arbitrary distribution of population probability P_m is given for the oscillator levels $m_{\max}$ ($m_{\max}$ is the quantity of discrete quantum states taken into account) (naturally, $\sum_{m=0}^{m_{\max}} P_m = 1$). Then, accidentally, some level number n is chosen under the assumption that the transition is possible from it, and the change in P_m ($m = 0, 1, \ldots, m_{\max}$) is calculated.

In case of a large number of oscillators in some ensemble, the prerequisites for the oscillator output from the equilibrium position do not depend on the number of the "filled" state; and they are formed not for all ensemble oscillators in the nth energy level. Therefore, accounting for the Bayes theorem,[15] the probability change ΔP_m of the mth quantum population level is calculated by the formula

$$\Delta P_m = \begin{cases} \nu P_n Q_{nm} \ (m \neq n) \\ \nu P_n (Q_{nn} - 1) \ (m = n), \end{cases}$$

where ν is the uniformly distributed random number within [0, 1].

The found values of ΔP_m are added to P_m, which results in a new distribution, for which the same procedure is repeated. Summing up the obtained values of P_m for each level, after a rather large number of iterations, the mean value of $\langle P_m \rangle$ is calculated for the population probability of each energy level:

$$\langle P_m \rangle = \frac{\sum_{i=1}^{It} (P_m)_i}{It}, \quad m = 0, 1, 2, \ldots, m_{\max},$$

It is the number of iterations; $(P_m)_i$ is the current value of population probability for the mth level.

The distribution obtained as a result of these calculations is the sought one, averaged over the oscillator ensemble by the probability distribution of the quantum level population of the oscillator interacting with the environment.

Results of mathematical experiment

Figures 1–3 present the results of numerical modeling by means of the above-described algorithm for different values of $y = \hbar\omega/(kT)$. The found probability distribution of the energy level population is shown by the circles; a solid line with small squares denotes the Boltzmann distribution is presented for comparison. All distributions are resulted from averaging of 10^7 ensembles calculated at $m_{\max} = 200$.

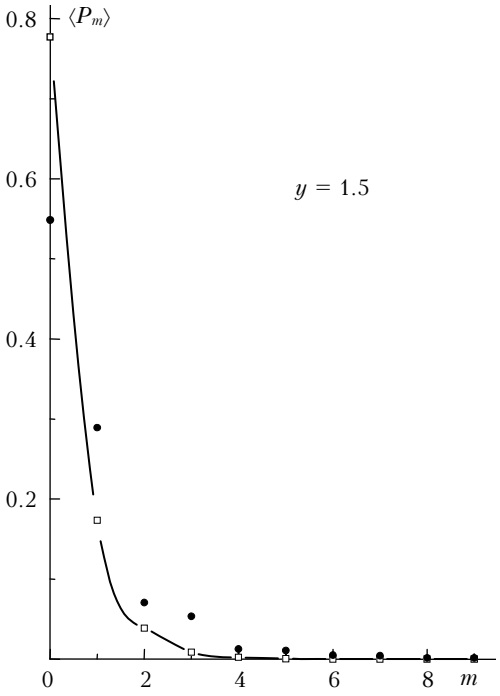

Fig. 1.

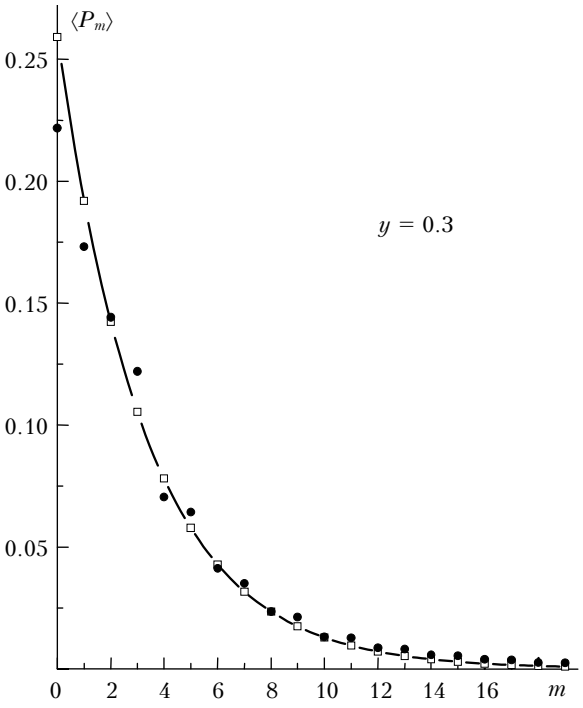

Fig. 2.

As it follows from Figs. 1–3, the permanent stochastic disturbance and interaction of oscillators with the environment leads to much higher population probabilities for the oscillators of lower levels than for upper ones. Moreover, the found probability distributions are close enough to the Boltzmann distribution. Note that numerical modeling at $y > 2$ has shown, as it was expected, the Bose-condensation of the oscillator states.

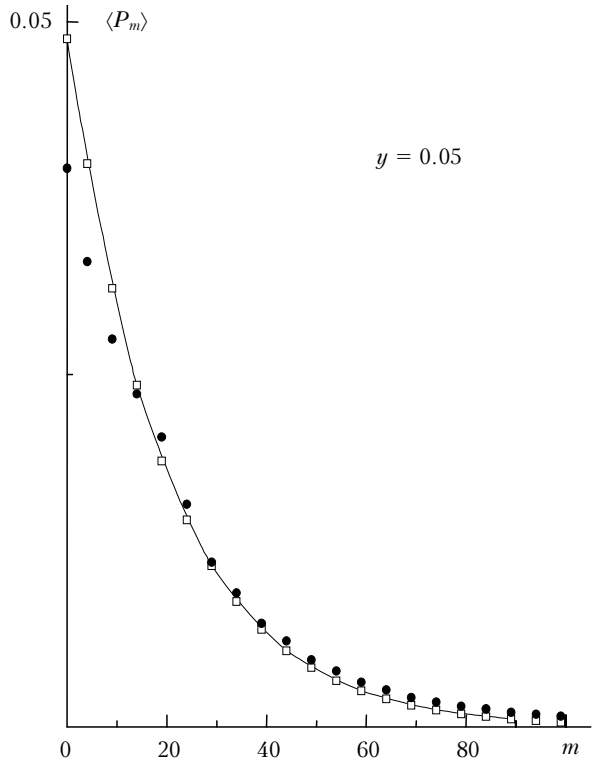

Fig. 3.

Transitions from upper quantum levels to low ones depend on the electromagnetic energy radiation. Therefore, the above mechanism of energy levels population allows one at the set value of y to determine the spectral distribution $I(\Delta n)$ for spontaneous radiation of the weakly connected oscillators (Δn is the difference between level numbers). In this case it is sufficient to estimate the contribution into the radiation energy of each possible transition from the upper energy level to low one. Figures 4–6 present the corresponding dependences calculated at the same parameters as in Figs. 1–3.

As it follows from the presented dependences, at relatively high values of y, the main energy is emitted at transitions between close levels. However, as the temperature grows, the radiation maximum is shifted closer to the short-wave region. In addition, based on the numerical experiment data for the wavelength of the radiation maximum, one can write down the approximated empirical relation: $\lambda = \beta/T$, where $\beta = 1.7 \cdot 10^{-2}$ m·K.

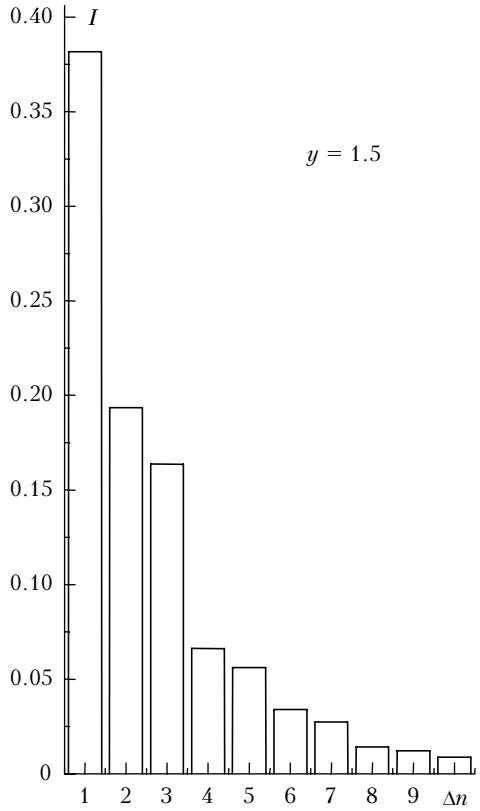

Fig. 4.

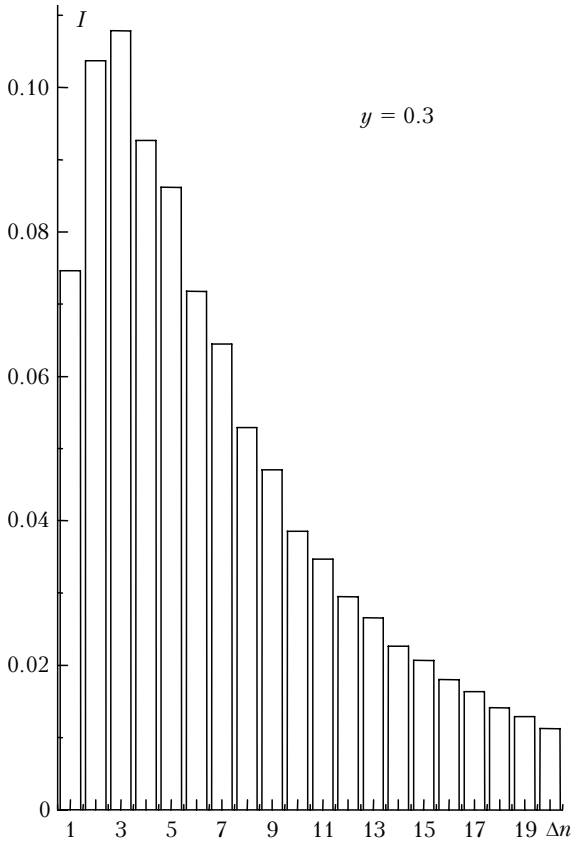

Fig. 5.

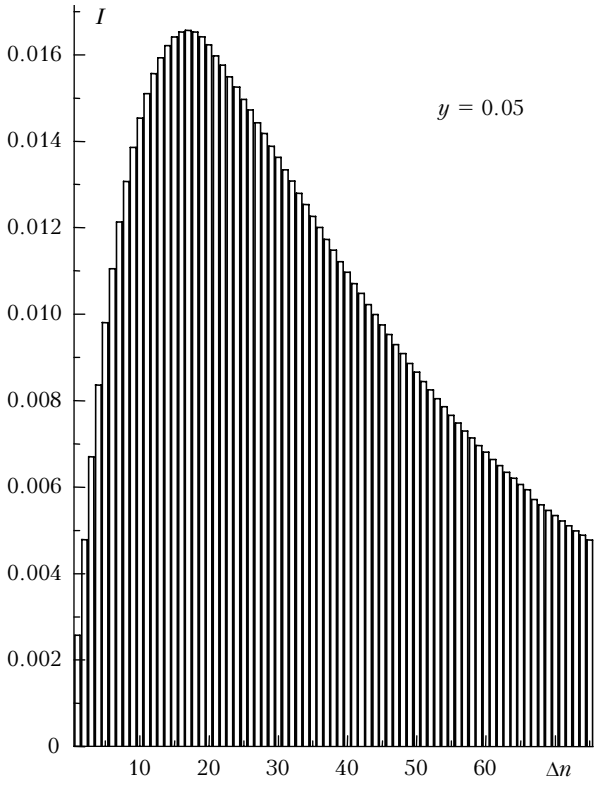

Fig. 6.

Conclusion

Thus, as follows from the obtained results, the permanent stochastic disturbance and interaction of oscillators with the environment are the main reasons for the physical systems close to oscillators by its properties, that they are in the states coinciding with eigenfunctions of the stationary Schrödinger equation, and their transitions from state to state occur in the form of a jump. The effect of these factors causes a higher population probability for the lower energy levels than for upper ones.

In conclusion, let us note a consequence of Bose-condensation of oscillator states, subjected to the permanent stochastic disturbance, which can have an applied significance and requires an experimental verification. The condensation does not depend on whether oscillators collide with each other or with other particles, for example, with electrons. It causes the ceasing of energy exchange between the environment and vibrational degrees of freedom of oscillators. This circumstance points out to the fact that substances with free charge carriers and close to the system of weakly connected oscillators by properties, when traversing by the electric current at a temperature below $T = \hbar\omega/(2k)$, should have the minimal energy for heating (ω is the frequency of natural vibrations of the oscillator material).

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