SERIES EXPANSIONS OF THE COHERENCE FUNCTION FOR USE IN INVERSE PROBLEMS OF THE RADIATION TRANSFER THEORY IN A SMALL ANGLE APPROXIMATION

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Numerical techniques are considered for reconstructing spatial coherence function from angular distributions of the intensity of multiply scattered plane wave using the small angle approximation of the radiation transfer theory based on the use of Fourier–Bessel and Dini series expansions. Accuracy of the techniques is estimated in numerical experiments with the model function for which the analytical form of Hankel transform is known.

STATEMENT OF THE PROBLEM

When solving inverse problems of the radiation transfer theory in a small angle approximation, the reconstruction of the spatial coherence function $F(\rho)$ from the angular distribution of the intensity of multiply scattered plane wave[1] $I(\omega)$,

$$\frac{k^2}{2\pi}\int_0^{2R} \rho J_0(\omega\rho)F(\rho)\mathrm{d}\rho = I(\omega), \qquad (1)$$

is an important point. In Eq. (1) $J_0(x)$ is the zeroth order Bessel function: $\omega = k\sin\theta$ is the spatial frequency; $k = 2\pi/\lambda$; and, θ is the scattering angle. In turn, the data on the coherence function of scattered field $F(\rho)$ related to the correlation function of the particle shadow $\varphi(\rho)$ and the optical thickness of the medium by the relationship[2]

$$F(\rho) = \exp[-\tau + \tau\varphi(\rho)/2] - \exp(-\tau) \qquad (2)$$

allow one to determine small angle scattering phase function, optical transfer function, and point spread function of the medium, as well as its disperse composition.

Peculiarity of the problem is that the intensity $I(\omega)$ is normally measured at discrete points ω_j within a finite interval $[\omega_{\min}, \omega_{\max}]$. On the one hand, limitation of the minimum frequency $\omega_{\min}$ is connected with the practical difficulties in separation of direct and scattered radiation near the forward direction; on the other hand, the role of noise component significantly increases at high ω, when the small-angle intensity $I(\omega)$ comes to zero. This component is caused both by the measurement errors and by the contribution from scattered radiation which is not allowed for in the small-angle approximation given by Eqs. (1) and (2). The choice of maximum frequency $\omega_{\max}$ is conditioned by the necessity of decreasing the effect from these errors.

Under conditions when the experimental data set is not complete, it is expedient to take into account *a priori* information on the solution sought for reconstructing the coherence function $F(\rho)$ from the angular distribution of the intensity $I(\omega)$. Such an information may be continuity, monotony, and downward convexity and boundedness of the coherence function $F(\rho)$ on the interval $0 \le \rho \le 2R$:

$$0.25 \ge F(0) \ge F(\rho) \ge 0, \quad F'(\rho) \le 0, \quad F''(\rho) \ge 0. \qquad (3)$$

Vagin and Veretennikov[1] have proposed the finite-difference algorithms for inverting Eq. (1) based on minimization of the discrepancy functional on the set of functions satisfying the limitations (3). In this paper we consider the techniques for reconstructing the coherence function $F(\rho)$ from Eq. (1) based on the expansion of the functions with limited spectrum into Fourier–Bessel and Dini series. It is essential for the approach proposed that the coherence function $F(\rho)$ is equal to zero outside the interval $0 \le \rho \le 2R$.

METHOD OF SOLUTION

Let us consider the integral relationship[3] known in the Bessel function theory

$$\int_0^1 x J_0(\alpha_i x)\, J_0(\alpha_j x)\, \mathrm{d}x = \begin{cases} \frac{1}{2}\, J_1^2(\alpha_i), & i = j, \\ 0, & i \ne j, \end{cases} \qquad (4)$$

where α_i are the zeros of the Bessel function $J_0(x)$: $J_0(\alpha_i) = 0$, $I = 1, 2, \ldots$. Orthogonality of the system of functions $\{J_0(\alpha_i x)\}$ with the weight x on the interval (0, 1) follows from Eq. (4). The functions $J_0(\alpha_i x) = \sqrt{2}/J_1(\alpha_i x)$ are normalized on the interval (0, 1).

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The functions $J_0(\alpha_i x)$, $i = 1, 2, \ldots$ form a complete system that allows one to write the expansion of an arbitrary function $f(x)$ with limited variation within the interval(0, 1) into the Fourier–Bessel series:

$$f(x) = \sum_{i=1}^{\infty} \alpha_i J_0(\alpha_i x), \quad \text{at } 0 < x < 1 , \tag{5}$$

with the coefficients

$$\alpha_i = \frac{2}{J_1^2(\alpha_i)} \int_0^1 x J_0(\alpha_i x) f(x)\, dx. \tag{6}$$

If the function $f(x)$ is continuous on the interval (0, 1) and the product $\sqrt{x}\, f(x)$ is absolutely integrable, then the Fourier–Bessel series of the function $f(x)$ given by Eq. (5) uniformly converges on the interval (0, 1) (Ref. 3).

The coherence function $F(\rho)$ is determined on the interval [0, 2R] and satisfies all the above conditions. So one can expand it into the Fourier–Bessel series

$$F(\rho) = \frac{\pi}{(kR)^2} \sum_{i=1}^{\infty} \frac{J_0(\omega_i \rho)}{J_1^2(\alpha_i)}) \, I(\omega_i), \quad 0 < \rho < 2R, \tag{7}$$

where $\omega_i = \alpha_i / 2R$; $I(\omega_i)$ is the intensity at the point $\omega = \omega_i$. Formula (7) expresses the coherence function $F(\rho)$ in terms of its Hankel transform $I(\omega)$ at the points $\omega_i = \alpha_i / 2R$, $i = 1,2, \ldots$, where α_i are the zeros of the Bessel function $J_0(x)$, and can be considered as the formula for inverting the integral equation (1). It follows from Eq. (7) that the information on the intensity distribution $I(\theta)$ near the forward direction, for which $\theta < \alpha_1 / 2kR \cong (kr)^{-1}$, is not required for reconstructing the function $F(\rho)$.

For reconstructing the function $F(\rho)$ by formula (7) it is necessary to *a priori* set the maximum radius of particles R, on which the position of the nodes $\omega_i = \alpha_i / 2R$ depends. The errors in choosing R can lead to violation of the $F(\rho)$ analytical properties determined by the relationships (3). Therefore examination of the conditions (3) for the solution reconstructed is a criterion of the correctness in the *a priori* assignment of R values.

Summing of the series (7) is unstable with respect to the errors in setting $I(\omega_i)$, if the error in reconstructing $F(\rho)$ is estimated in the uniform metrics. Stable techniques for summing the Fourier series based on the regularization are described in Ref. 4. In the simplest case, the limitation of the number of terms in the expansion (7), corresponding to the errors in $I(\omega_i)$ setting, serves for this purpose. Substituting the expansion (7) into Eq. (1), we obtain the following formula

$$I(\omega) = \frac{1}{R} \sum_{i=1}^{\infty} \frac{I(\omega_i)\, \omega_i J_0(2R\omega)}{J_1(\alpha_i)\left[\omega_i^2 - \omega^2\right]} , \tag{8}$$

which means that the function $I(\omega)$ can be reconstructed from its values at the points $\omega_i = \alpha_i / 2R$, $i = 1, 2, \ldots$.

Let us now consider the partial sum $F_n(\rho)$ corresponding to the finite number n of terms in the expansion (7). The partial sum $I_n(\omega)$ of the series (8) corresponds to it. At the limit transition $\omega \to \omega_j$ in Eq. (8), we obtain for $I_n(\omega)$:

$$I_n(\omega_j) = \begin{cases} I(\omega_j), & j \le n; \\ 0, & j > n. \end{cases} \tag{9}$$

It follows from Eq. (9) that the discrepancy of the Eq. (1) determined at the set of points ω_j, $j = 1, \ldots, n$, is equal to zero for any finite number of the terms of the expansion (7). If the value

$$\sigma^2 = \int_0^{\infty} [I(\omega) - I_n(\omega)]^2 \omega d\omega, \tag{10}$$

is taken as a measure of deviation of the functions $I(\omega)$ and $I_n(\omega)$, then we obtain by the Parceval formula:

$$\sigma^2 = \left(\frac{k^2}{2\pi}\right)^2 \int_0^{2R} [F(\rho) - F_n(\rho)]^2 \rho d\rho = \frac{1}{2R^2} \sum_{i=n+1}^{\infty} \frac{I^2(\omega_i)}{J_1^2(\alpha_i)}. \tag{11}$$

As follows from Eq. (11) the discrepancy of Eq. (1) is determined in the metrics L_2, when replacing the series (7) by the finite sum of n terms, by the values of the function $I(\omega)$ at the points $\omega_i = \alpha_i / 2R$, $i > n$.

It should be noted that one can construct other Fourier series of the function $F(\rho)$, the coefficients in which are determined by its Hankel transform, i.e. by the intensity $I(\omega)$. For example, if the roots of the equation $J_1(x) = 0$ $i = 1, 2, \ldots$ are β_i, then we obtain a new system of functions $J_0(\beta_i x)$ that are orthogonal on the interval (0, 1) with the weight x:

$$\int_0^1 x J_0(\beta_i x) J_0(\beta_j x) dx = \begin{cases} \frac{1}{2} \; J_0^2(\beta_i), & i = j, \\ 0, & i \ne j, \end{cases} \tag{12}$$

Expansion of an arbitrary function $f(x)$ into a series over the system of functions $\{J_0(\beta_i x), i = 1, 2, \ldots\}$ has the form analogous to Eq. (5) where α_i is replaced by β_i and with the coefficients

$$b_i = \frac{2}{J_0^2(\beta_i)} \int_0^1 x \, J_0(\beta_i x) \, f(x) dx , \tag{13}$$

and one can write the series for the function $F(\rho)$

$$F(\rho) = \frac{\pi}{(kR)^2} \sum_{i=1}^{\infty} \frac{J_0(\omega_i \rho)}{J_0^2(\beta_i)}) \, I(\omega_i), \quad 0 < \rho < 2R, \quad (14)$$

where $\omega_i = \alpha_i / 2R$, $i = 1, 2, \ldots$. One should recognize that the expansion (14) is less correct in comparison with the expansion (7), because the first zero of the function $J_1(x)$ $\beta_1 = 0$ and it is necessary to set $I(\omega_1) = I(0)$ in Eq. (14).

Finally, in order to invert the integral equation (1) one can use the expansion into the Dini series.[3] Let us consider the function $p(x) = q\, J_0(x) - x\, J_1(x)$, where q is an arbitrary constant. It is known that for any q the function $p(x)$ has a root between any two roots of the function $J_0(x)$. Let γ_1, γ_2, ..., γ_i ... be the roots of the equation

$$qJ_0(x) - x\, J_1(x) = 0. \quad (15)$$

It can be shown that the functions $J_0(\gamma_i\, x)$, $i = 1, 2, \ldots$ are orthogonal on the interval (0, 1) with the weight x, and the condition of the function $J_0(\gamma_i\, x)$ normalization has the form:

$$\int_0^1 x\, J_0^2(\gamma_i\, x)\mathrm{d}x = \frac{1}{2}\left[J_0^2(\gamma_i) + J_1^2(\gamma_i)\right] = \frac{1}{2} J_0^2(\gamma_i)\left[1 + \frac{q^2}{\gamma_i^2}\right]. \quad (16)$$

The Dini series for the function $f(x)$ on the interval (0, 1) has the form of Eq. (5) with α_i replaced by β_i and with the coefficients:

$$c_i = \frac{2}{J_0^2(\gamma_i)} \frac{\gamma_i^2}{q^2 + \gamma_i^2} \int_0^1 x\, J_0(\gamma_i\, x)\, f(x)\mathrm{d}x \, . \quad (17)$$

The Dini series for the function $F(\rho)$ has the form:

$$F(\rho) = \frac{\pi}{(kR)^2} \sum_{i=1}^{\infty} h_i\, J_0(\omega_i\, \rho)\, I(\omega_i), \quad (18)$$

$0 < \rho < 2R$, where $\omega_i = \gamma_i / 2R$; and γ_i are the roots of Eq. (15);

$$h_i = \frac{1}{J_0^2(\gamma_i)} \frac{\gamma_i^2}{q^2 + \gamma_i^2} = \frac{1}{J_1^2(\gamma_i)} \frac{q^2}{q^2 + \gamma_i^2} \, . \quad (19)$$

The important property of the expansion (18) is the possibility of varying of the left boundary $\omega_{\min}$ in the region where it is necessary to measure $I(\omega)$, using the corresponding choice of the parameter q in Eq. (15).

RESULTS OF THE NUMERICAL SIMULATIONS

Model calculations have been carried out in order to study the efficiency of using series expansions for inverting Eq. (1). The function $G(x)$,

$$G(x) = \begin{cases} \frac{2}{\pi}\left[\arccos x - x\sqrt{1 - x^2}\right], & x \le 1, \\ 0, & x > 1, \end{cases} \quad (20)$$

was considered as the exact solution of Eq. (1). Its Hankel transform has the known analytical form:

$$\int_0^1 x\, J_0(x\omega)\, G(x)\, \mathrm{d}x = 2\left[\frac{J_1(\omega/2)}{\omega}\right] \equiv I(\omega) \, , \quad (21)$$

and is the normalized small-angle scattering phase function in the Fraunhofer diffraction approximation. The function $G(x)$ is reconstructed by the formula

$$f(x) = \frac{2}{R^2} \sum_{i=1}^{\infty} \frac{J_0(\omega_i\, x)}{J_0^2(\beta_i)}) \, I(\omega_i), \quad 0 < x < R, \quad (22)$$

where $\omega_i = \beta_i / R$, and β_i are the zeros of the function $J_1(x)$.

The consequences of replacing the series (22) by the finite sum, errors in setting $I(\omega_i)$, and *a priori* choice of the parameter R were studied in the numerical experiments. The choice of R is connected with estimating the upper limit of integration in Eq. (21). Obviously, the errors in reconstructing $G(x)$ by Eq. (22) can appear in the case of Eq. (21) for $R < 1$ and they are determined by the contribution from the integral $\int_R^1 x J_0(x\omega)\, G(x)\, \mathrm{d}x$ into the value of $I(\omega)$. The relative errors in setting $I(\omega_i)$ have been modeled by a random law with a uniform distribution on the interval $[-\varepsilon, \varepsilon]$.

Table I gives the information on the convergence of the partial sums of the series (22), where n is the number of terms in the expansion, ε_G is the relative root-mean-square error in reconstructing $G(x)$. As is seen from the table, high accuracy of the approximation of $G(x)$ (Eq. (20)) by the series (22) is reached for the number of expansion terms $n = 10$ ($\varepsilon_G = 2.2\%$), i.e. for reconstructing $G(x)$ it is enough to have the data on the value of $I(\omega)$ at only 10 points within the interval $0 < \omega \le \beta_{10} \simeq 32.19$. Then there is a significant gain in the rate of operation in comparison with the techniques considered in Ref. 1. The series (22) converges worse in the vicinity of the point $x = 0$, for which the relative errors ε_0 in reconstructing $G(0)$ are given in Table I. It leads to violation of the convexity of the function $G(x)$ in the vicinity of $x = 0$, that can be corrected based on the *a priori* information on the solution determined by Eq. (3).

The effect of the error ε in setting $I(\omega)$ on the error in reconstructing ε_G for different n is also shown in Table I. The results presented in the third row indicate the stability of summing Eq. (22) relative to the error in the initial data. It is seen from the table that, starting from $n = 10$, further increase in the number of terms in the expansion (22) does not result

in increase of the accuracy of reconstruction of the model function $G(x)$ at the presence of errors in the initial data, and even small increase in the error ε_G is observed for $n \geq 40$. For $n = 50$ the dependence of ε_G on ε is practically linear with the "error amplification coefficientB $\varepsilon_G/\varepsilon \simeq 0.6$ in the interval $1 \leq \varepsilon \leq 10\%$. Random errors in the initial data show a stronger effect on the accuracy of reconstruction of $G(x)$ at $x \to 1$.

TABLE I.

N	n		3	5	10	15	20	40	50
1	ε_G, %	$\varepsilon = 0$	10	5.2	2.2	1.6	1.1	0.6	0.4
2	ε_0, %	$R = 1$	17	9.2	4.0	2.8	2.0	1.0	0.8
3	ε_G, %	$\varepsilon = 10\%$, $R = 1$	–	6.2	5.5	–	5.5	5.7	5.8
4	ε_G, %	$\varepsilon = 10\%$, $R = 1.5$	–	8.1	3.6	–	2.4	2.1	–

A family of functions $f(x)$ reconstructed by formula (22) with the exact initial data ($\varepsilon = 0$), $n = 50$ and different values $R \leq 1$ is shown in Fig. 1. Curve *1* corresponds to $R = 1$ and practically coincides with the exact solution $G(x)$. Other curves, *2* to *5*, are obtained with the sequence of decreasing values $R = 0.9$, 0.8, 0.7, and 0.6. The examples shown in Fig. 1 illustrate the effect of errors in choosing R on the shape of the solution reconstructed. Since Eq. (22) is the expansion of the function $G(x)$ on the interval $(0, R)$, then each curve shown in Fig. 1 should also be considered on the relevant intervals, for example, curve *2* is on the interval $0 < x < 0.9$, etc.

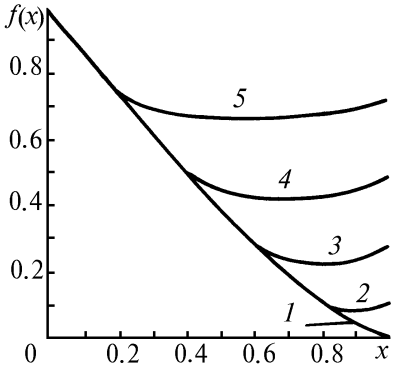

FIG. 1. Reconstruction of the model function $G(x)$ (Eq. (20)) by the technique of summing the Fourier–Bessel series (22) for $n = 50$, $R = 1$ (1), 0.9 (2), 0.8 (3), 0.7 (4), and 0.6 (5) and exact setting of $I(\omega)$.

One can note that for the fixed value R each of the reconstructed functions $f(x)$ keeps its monotonic decrease within the interval $(0, R)$, reaches its minimum at the boundary $x = R$, and then significantly increases at $R < x \leq 1$. If we designate $\Delta = 1 - R$ as an error in choosing R, then the function $f(x)$ satisfactorily approximates the function $G(x)$ on the interval $0 \leq x < R-\Delta$; and their divergence is observed to the right of the point $x = R-\Delta$. The width of this interval vanishes at $R \to 1/2$, that determines the lower boundary of the acceptable values R equal to the half of the true value $R = 1$.

Thus, the choice of $R < 1$ essentially influences the deviation of $f(x)$ from $G(x)$. The rms error in reconstructing $G(x)$ on the interval (0, 1) is shown in Fig. 2 as a function of R. It is seen from Fig. 2 that the behavior of ε_G is not symmetrical relative to the point $R = 1$, and the choice of $R > 1$ practically does not result in a poorer quality of the reconstruction. One should take into account these peculiarities in the behavior of the error $\varepsilon_G(R)$ when choosing R *a priori*.

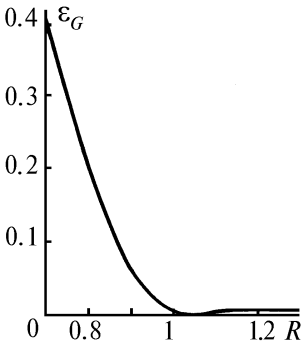

FIG. 2. Relative error in reconstructing the model function $G(x)$ by the technique of summing the Fourier–Bessel series on the interval (0, 1) as a function of R.

Let us note in conclusion a peculiarity that appears when choosing $R > 1$. As is seen from the bottom row of the Table I, the choice of $R > 1$ at presence of errors ($\varepsilon \neq 0$) results in a decrease in the error in reconstructing the function $G(x)$ for $n = 10$ and higher. In contrast to the case of the exact setting R (third row), the increase of n up to 40 is accompanied by a monotonic decrease in the error ε_G.

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