ON THE THEORY OF OPTICAL TRANSFER OPERATOR OF THE ATMOSPHERE–OCEAN SYSTEM

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Received March 9, 1995

The optical transfer operator (OTO) of the atmosphere–ocean system (AOS) as a three-dimensional plane layer with the horizontally inhomogeneous reflecting and refracting interface between two media is formulated by the method of influence functions (IFs) and spatial-frequency characteristics (SFCs) using the perturbation theory series. The case is treated first when no splitting into spatial and angular variables is used for the refraction and reflection coefficients. Such an OTO has the most general form and is expressed in terms of the linear IF and SFC of the atmosphere and the ocean. The solution of the problem for the entire AOS is reduced to the solution of two problems for each medium.

INTRODUCTION

In this paper, we present the mathematical models formulated for detailed description of the process of radiative field formation and image transfer in the atmosphere–ocean system (AOS) with a horizontally inhomogeneous interface. The solar radiation propagation in the AOS is commonly described by the boundary-value problem of the transfer theory for a layer with a nonorthotropic interface between two media, when the ocean is modeled as a reflecting underlying surface. Generalized solutions of such problem in the form of linear and nonlinear functionals, with the universal characteristics of the linear transfer system being their kernels, are constructed by the method of the influence functions (IFs) and spatial frequency characteristics (SFCs).[1–8]

The functions so constructed establish explicit relations between problem solution and the characteristics of sources and the reflecting interface, as well as outline a "scenario" at the interface with regard to the contribution of multiple scattering in a medium and multiple re-reflection from the interface, and the scenario transfer through a turbid reflecting and absorbing medium. These functionals describe the optical transfer operator (OTO) of the atmosphere–underlying surface system, including the case in which spatial and angular dependences in the reflection operators cannot be factorized.[5–8]

The problems for the AOS with the reflecting and transmitting interface are more complicated. For the case of a horizontally homogeneous smooth or wavy interface, the numerical algorithms have been developed for modeling the radiation in the AOS,[2,9] and the OTO has been formulated by the method of the influence functions.[10,11] We have developed the method of the IFs and SFCs as applied to two-medium problems with the horizontally inhomogeneous interface when spatial and angular dependences cannot be separated for the reflection and transmission operators.[12]

The perturbation theory series and the theory of generalized solutions of kinetic equations provide the basis for mathematical apparatus of the IF, SFC, and OTO models. Here we present new results demonstrating how the solution can be obtained for any order of multiplicity of radiation interaction with the interface and the OTO of AOS can be constructed with regard to multiple scattering in each medium with the help of universal linear transfer characteristics: the IFs of the atmosphere (Θ_a) and the ocean (Θ_{oc}) or corresponding SFCs of the atmosphere (ψ_a) and the ocean (ψ_{oc}).

We have derived the most general representation of the OTO of AOS, from which a particular representation in any (linear and nonlinear) approximations can be found. The reduction of the solution of one boundary-value problem of transfer for the two-medium AOS to the solution of two boundary-value problems for each medium separately and the formulation of the OTO in the matrix form with the two-component vector of IF $\mathbf{\Theta} = \{\Theta_a, \Theta_{oc}\}$ or SFC $\boldsymbol{\psi} = \{\psi_a, \psi_{oc}\}$ as a kernel are crucially new results in the approach proposed here. For the first time this approach was proposed by us in Refs. 2 and 10–12.

The mathematical models of IF, SFC, and OTO constructed here allow one to develop new algorithms for remote sensing of the AOS, theory of vision and image transfer in a turbid media, as well as numerical simulation of radiation fields in the AOS illuminated by the Sun or other source.

0235-6880/96/01 17-10 $02.00

FORMULATION OF THE PROBLEM

Let us consider a plane layer infinite in horizontal direction ($-\infty < x, y < \infty$) of finite height ($0 \le z \le H$), which is illuminated in any way (from the top, from the bottom, or from the inside). At the level $z = h$ within the layer, the interface between two media is located, which reflects and transmits the radiation. The underlying surface is at the bottom ($z = H$). The system atmosphere–interface–ocean–bottom is considered as nonmultiplicative (without multiplication).

The direction of radiation propagation is specified by vector $\mathbf{s} = (\mu, \varphi)$, $\mu = \cos\vartheta$, $\mu \in [-1,1]$ on unit sphere $\Omega = [-1,1]\times[0, 2\pi]$, where $\vartheta \in [0, 180°]$ is the zenith angle counted off from the positive direction of the z axis, $\varphi \in [0, 2\pi]$ is the azimuth angle counted off from the x axis. The value $\varphi = 0$ is in the plane of solar vertical, being coincident with the XOZ plane. The solar flux is incident on the layer boundary $z = 0$ in the direction $\mathbf{s}_0 = (\mu_0, \varphi_0)$ with the zenith angle $\vartheta_0 \in [0, 90°]$, $\mu_0 = \cos\vartheta_0$, and the azimuth angle $\varphi_0 = 0$. For downward transmitted radiation we introduce the hemisphere of directions $\Omega^+ = \{(\mu, \varphi): \mu > 0\}$, and for upward reflected radiation - the hemisphere $\Omega^- = \{(\mu, \varphi): \mu < 0\}$, $\Omega = \Omega^+ \cup \Omega^-$.

The boundary conditions are written using the following sets:

$$\Gamma_0 = \{(z, r_\perp, s): z = 0,\ s \in \Omega^+\},$$
$$\Gamma_H = \{(z, r_\perp, s): z = H,\ s \in \Omega^-\},$$
$$\Gamma_h^+ = \{(z, r_\perp, s): z = h,\ s \in \Omega^+\},$$
$$\Gamma_h^- = \{(z, r_\perp, s): z = h,\ s \in \Omega^-\}.$$

The radiation passage through the interface is described by the reflection ($\hat{R}_1$ and $\hat{R}_2$) and transmission ($\hat{T}_{12}$ and $\hat{T}_{21}$) operators, where subscript *1* is for the upper layer (usually, the atmosphere), and subscript *2* is for the lower layer (the ocean):

$$[\hat{R}_1 \Phi](h, r_\perp, s) = \int_{\Omega^+} \Phi(h, r_\perp, s^+) q_1(r_\perp, s, s^+) \mathrm{d}s^+,\quad s \in \Omega^-, \tag{1}$$

$$[\hat{R}_2 \Phi]\ (h, r_\perp, s) = \int_{\Omega^-} \Phi(h, r_\perp, s^-)\ q_2(r_\perp, s, s^-)\ \mathrm{d}s^-,\quad s \in \Omega^+, \tag{2}$$

$$[\hat{T}_{12} \Phi]\ (h, r_\perp, s) = \int_{\Omega^+} \Phi(h, r_\perp, s^+)\ t_{12}(r_\perp, s, s^+)\ \mathrm{d}s^+,\quad s \in \Omega^+, \tag{3}$$

$$[\hat{T}_{21} \Phi]\ (h, r_\perp, s) = \int_{\Omega^-} \Phi(h, r_\perp, s^-)\ t_{21}(r_\perp, s, s^-)\ \mathrm{d}s^-,\quad s \in \Omega^-. \tag{4}$$

Optical properties of the atmosphere and the ocean are described by the altitude profiles of the total extinction coefficient $\sigma_t(z) = \sigma_s(z) + \sigma_{abs}(z)$, absorption coefficient $\sigma_{abs}(z)$, total scattering coefficient $\sigma_s(z) = \sigma_a(z) + \sigma_m(z)$, which includes the aerosol (hydrosol) $\sigma_a(z)$ and molecular $\sigma_m(z)$ components, and the total scattering phase function (χ is the scattering angle):

$$\gamma(z, \chi) = \frac{\sigma_a(z)\ \gamma_a(z, \chi)}{\sigma_s(z)} + \frac{\sigma_m(z)\ \gamma_m(\chi)}{\sigma_s(z)},$$

which in general comprises the aerosol (hydrosol) $\gamma_a(z, \chi)$ and molecular (Rayleigh) $\gamma_m(\chi) = 3(1-\cos^2 x)/(16\pi)$ components.

The integral operator of the kinetic equation $\hat{K} \equiv \hat{D} - \hat{S}$ includes the collision integral

$$\hat{S}\Phi \equiv \sigma_s(z) \int_\Omega \Phi\gamma\, \mathrm{d}s'$$

and the transfer operator

$$\hat{D} \equiv (s, \mathrm{grad}) + \sigma_t(z) = \hat{D}_z + \sin\vartheta \cos\varphi \frac{\partial}{\partial x} + \sin\vartheta \sin\varphi \frac{\partial}{\partial y}.$$

In 1D case,

$$\hat{K}_z \equiv \hat{D}_z - \hat{S},\ \hat{D}_z \equiv \mu \frac{\partial}{\partial z} + \sigma_t(z).$$

ON THE SEPARATION INTO CONTRIBUTIONS FROM THE ATMOSPHERE AND THE OCEAN

The radiation propagation in the atmosphere–ocean system with the interface is described by the general boundary-value problem of the transfer theory[2]:

$$\begin{cases} \hat{K}\Phi = F_b,\ \Phi|_{\Gamma_0} = f_0,\ \Phi|_{\Gamma_H} = f_H + \hat{R}_H \Phi, \\ \Phi|_{\Gamma_h^+} = \hat{R}_2 \Phi + \hat{T}_{12} \Phi + f_h^+,\ \Phi|_{\Gamma_h^-} = \hat{R}_1 \Phi + \hat{T}_{21} \Phi + f_h^-, \end{cases} \tag{5}$$

where F_b, f_0, f_H, f_h^+, and f_h^- are possible sources of radiation. A single event of radiation interaction with reflecting bottom is described by the operator

$$[\hat{R}_H \Phi](H, r_\perp, s) \equiv \int_{\Omega^+} \Phi(H, r_\perp, s^+)\ q_H(r_\perp, s, s^+)\, \mathrm{d}s^+. \tag{6}$$

Here, we do not detail the reflection and transmission operators, and use only their general form.

Because the boundary-value problem (5) is linear with respect to sources, the net radiation field of the system can be considered as superposition of solutions to a set of boundary-value problems of type (5) with one of the radiation sources F_b, f_0, f_H, f_h^+, and f_h^-, respectively. Using the case in which the system is illuminated by the solar flux as an example, we show how the initial boundary-value problem of the transfer theory can be separated into the boundary-value problems for the components of the light field: $\Phi = \Phi^0 + \Phi_a + \Phi_{aR} + \Phi_{oc} + \Phi_q$.

In the case of system illumination by the solar flux ($f_H = F_b = f_h^+ = f_h^- = 0$), the direct attenuated radiation Φ^0 is sought as the solution to the Cauchy problem:

$$\left\{\hat{D}_z \Phi^0 = 0, \quad \Phi^0|_{\Gamma_0} = \pi S_\lambda \delta(s - s_0), \quad \Phi|_{\Gamma_h^-} = 0 \right. \qquad (7)$$

for the upper layer $z \in [0, h]$ and $\Phi^0 \neq 0$ only for $s = s_0$.

The background radiation of the atmosphere Φ_a is sought as the solution of a 1D problem for a plane layer $z \in [0, h]$ with zero boundary conditions:

$$\left\{\hat{K}_z \Phi_a = \hat{S} \Phi^0, \quad \Phi_a|_{\Gamma_0} = 0, \quad \Phi_a|_{\Gamma_h^-} = 0. \right. \qquad (8)$$

The radiation of the atmosphere reflected from the interface is sought as the solution of the boundary-value problem for the layer $z \in [0, h]$ with the source at $z = h$:

$$\left\{\hat{K} \Phi_{aR} = 0, \ \Phi_{aR}|_{\Gamma_0} = 0, \ \Phi_{aR}|_{\Gamma_h^-} = \hat{R}_1 \Phi_{aR} + \hat{R}_1(\Phi^0 + \Phi_a). \right. \qquad (9)$$

In more detail, $\Phi_{aR} = \Phi_{aR}^0 + \Phi_{aR}^d$. The component Φ_{aR}^0 describes the contribution into the atmospheric haze due to scattering, within the upper layer, of the direct flux reflected from the interface ($z \in [0, h]$):

$$\left\{\hat{K} \Phi_{aR}^0 = 0, \ \Phi_{aR}^0|_{\Gamma_0} = 0, \ \Phi_{aR}^0|_{\Gamma_h^-} = \hat{R}_1 \Phi_{aR}^0 + \hat{R}_1 \Phi^0. \right. \qquad (10)$$

Because of atmospheric scattering of the haze diffuse component reflected from the interface, the component Φ_{aR}^d is formed, which is the solution of the problem ($z \in [0, h]$)

$$\left\{\hat{K} \Phi_{aR}^d = 0, \ \Phi_{aR}^d|_{\Gamma_0} = 0, \ \Phi_{aR}^d|_{\Gamma_h^-} = \hat{R}_1 \Phi_{aR}^d + \hat{R}_1 \Phi_a. \right. \qquad (11)$$

The radiation formed in the atmosphere is incident on the interface ($z = h$) and is the source of the component Φ_{oc} of the system light field (the ocean takes part in the formation of this component; $\Phi_{oc} \neq 0$ for $z \in [0; H]$):

$$\begin{cases} \hat{K}\Phi_{oc} = 0, \ \Phi_{oc}|_{\Gamma_0} = 0, \ \Phi_{oc}|_{\Gamma_H} = 0, \\ \Phi_{oc}|_{\Gamma_h^-} = \hat{R}_1 \Phi_{oc} + \hat{T}_{21} \Phi_{oc}, \\ \Phi_{oc}|_{\Gamma_h^+} = \hat{R}_2 \Phi_{oc} + \hat{T}_{12} \Phi_{oc} + \hat{T}_{12} (\Phi^0 + \Phi_a + \Phi_{aR}). \end{cases} \qquad (12)$$

For detailed consideration, the superposition

$$\Phi_{oc} = \Phi_{oc}^0 + \Phi_{oc}^d, \quad \Phi_{oc}^d = \Phi_{oc}^a + \Phi_{oc}^{aR},$$

may be introduced, in which the brightness field components caused by the influence of the direct solar radiation Φ_{oc}^0 (problem (12) with the source $\hat{T}_{12}(\Phi^0 + \Phi_{aR}^0)$) and the atmospheric haze Φ_{oc}^d (problem (12) with the source $\hat{T}_{12}(\Phi_a + \Phi_{aR}^d)$) are separated.

The contribution of illumination from the reflecting ocean bottom can be found as a solution of the boundary-value problem

$$\begin{cases} \hat{K}\Phi_q = 0, \ \Phi_q|_{\Gamma_0} = 0, \ \Phi_q|_{\Gamma_H} = \hat{R}_H \Phi_q + E_H, \\ \Phi_q|_{\Gamma_h^+} = \hat{R}_2 \Phi_q + \hat{T}_{12} \Phi_q, \ \Phi_q|_{\Gamma_h^-} = \hat{R}_1 \Phi_q + \hat{T}_{21} \Phi_q, \end{cases} \qquad (13)$$

in which the bottom illumination $E_H = \hat{R}_H \Phi_{oc}$ serves as a source.

EQUATIONS FOR INFLUENCE FUNCTIONS OF THE ATMOSPHERE AND THE OCEAN AND OPTICAL TRANSFER OPERATOR

The solutions of the 1D boundary-value problems of the transfer theory for direct solar radiation given by Eq. (7) and for atmospheric haze given by Eq. (8) are well known.[2] The 3D and 1D boundary-value problems given by Eqs. (9)–(11), in which the ocean was considered as a reflecting nonorthotropic or Lambertian surface, were studied in Refs. 1–8. The light field components Φ_{aR}^0, Φ_{aR}^d, and Φ_{aR} are calculated in terms of the influence function of the atmosphere $\Theta_a(s^-; z, r_\perp, s)$ being the solution of the boundary-value problem

$$\left\{\hat{K} \Theta_a = 0, \quad \Theta_a|_{\Gamma_0} = 0, \quad \Theta_a|_{\Gamma_h^-} = f_\delta (s^-; r_\perp, s) \right. \qquad (14)$$

with the source

$$f_\delta(s^-; r_\perp, s) \equiv \delta(r_\perp) \delta(s - s^-). \qquad (15)$$

The boundary-value problem for finding the individual components of radiation in the system formed under the effect of multiply scattered radiation in the ocean can be written in general as

$$\begin{cases} \hat{K}\Phi_{oc} = 0, \ \Phi_{oc}|_{\Gamma_0} = 0, \ \Phi_{oc}|_{\Gamma_h^-} = \eta(\hat{R}_1 \Phi_{oc} + \hat{T}_{21} \Phi_{oc} + E_a), \\ \Phi_{oc}|_{\Gamma_h^+} = \eta(\hat{R}_2 \Phi_{oc} + \hat{T}_{12} \Phi_{oc} + E_{oc}), \ \Phi_{oc}|_{\Gamma_H} = 0, \end{cases} \qquad (16)$$

where as sources of radiation serve the ocean illumination from the top (from the atmosphere) $E_{oc}(r_\perp, s)$ and the atmosphere illumination from the bottom (from the ocean) $E_a(r_\perp, s)$.

The solution to the problem (16) is sought as a perturbation series

$$\Phi_{oc}(z, r_\perp, s) = \sum_{n=1}^{\infty} \eta^n \Phi_n(z, r_\perp, s), \qquad (17)$$

with the parameter η indicating the act of radiation passage through the interface. Let us introduce the two-component vectors

$$\mathbf{\Phi}_n = \{\Phi_{a,n}, \Phi_{oc,n}\}, \ \ \mathbf{E} = \{E_a, E_{oc}\}, \ \ \mathbf{\Theta} = \{\Theta_a, \Theta_{oc}\}. \qquad (18)$$

In the linear approximation ($n = 1$), the boundary-value problem with two sources $E_{\rm a}$ and $E_{\rm oc}$

$$\begin{cases} \hat{K}\Phi_1 = 0, \ \ \Phi_1|_{\Gamma_0} = 0, \ \ \Phi_1|_{\Gamma_H} = 0, \\ \Phi_1|_{\Gamma_h^+} = E_{\rm oc}, \ \ \Phi_1|_{\Gamma_h^-} = E_{\rm a} \end{cases} \tag{19}$$

is separated into two boundary-value problems: for the ocean ($z\in[h, H]$)

$$\{\hat{K}\,\Phi_{{\rm oc},1} = 0, \quad \Phi_{{\rm oc},1}|_{\Gamma_H} = 0, \quad \Phi_{{\rm oc},1}|_{\Gamma_h^+} = E_{\rm oc} \tag{20}$$

and for the atmosphere ($z\in[0, h]$)

$$\{\hat{K}\,\Phi_{{\rm a},1} = 0, \quad \Phi_{{\rm a},1}|_{\Gamma_0} = 0, \quad \Phi_{{\rm a},1}|_{\Gamma_h^-} = E_{\rm a}. \tag{21}$$

Let us represent the illumination as functionals

$$E_{\rm a}(r_\perp, s) = \frac{1}{2\pi}\int\limits_{\Omega^-} \delta(s - s^-)\,{\rm d}s^- \int\limits_{-\infty}^{\infty} \delta(r_\perp - r'_\perp)\,E_{\rm a}(r'_\perp, s^-)\,{\rm d}r'_\perp, \tag{22}$$

$$E_{\rm oc}(r_\perp, s) = = \frac{1}{2\pi}\int\limits_{\Omega^+} \delta(s - s^+)\,{\rm d}s^+ \int\limits_{-\infty}^{\infty} \delta(r_\perp - r'_\perp)\,E_{\rm oc}(r'_\perp, s^+)\,{\rm d}r'_\perp, \tag{23}$$

then the solutions to problems (19) and (20) can be represented as linear functionals ($s\in\Omega$)

$$\Phi_{{\rm a},1}(z, r_\perp, s) = (\Theta_{\rm a}, E_{\rm a}) = = \frac{1}{2\pi}\int\limits_{\Omega^-} {\rm d}s_1^- \int\limits_{-\infty}^{\infty} \Theta_{\rm a}(s_1^-; z, r_\perp - r_{\perp 1}, s)\,E_{\rm a}(r_{\perp 1}, s_1^-)\,{\rm d}r_{\perp 1}, \quad z \in [0, h], \tag{24}$$

$$\Phi_{{\rm oc},1}(z, r_\perp, s) = (\Theta_{\rm oc}, E_{\rm oc}) = = \frac{1}{2\pi}\int\limits_{\Omega^+} {\rm d}s_1^+ \int\limits_{-\infty}^{\infty} \Theta_{\rm oc}(s_1^+; z, r_\perp - r_{\perp 1}, s)\,E_{\rm oc}(r_{\perp 1}, s_1^+)\,{\rm d}r_{\perp 1}, \quad z \in [h, H]. \tag{25}$$

The kernels of functionals (24) and (25) are the IF of the atmosphere $\Theta_{\rm a}(s^-, z, r_\perp, s)$, being the solution of boundary-value problem (14), and the IF of the ocean $\Theta_{\rm oc}(s^+; z, r_\perp, s)$, being the solution of the problem for the layer $z\in[h, H]$

$$\{\hat{K}\,\Theta_{\rm oc} = 0, \quad \Theta_{\rm oc}|_{\Gamma_H} = 0, \quad \Theta_{\rm oc}|_{\Gamma_h^+} = f_\delta(s^+; r_\perp, s) \tag{26}$$

with the source $f_\delta(s^+; r_\perp, s) \equiv \delta(r_\perp)\,\delta(s - s^+)$.

For the second and succeeding approximations ($n \ge 2$) of perturbation series (17), the boundary-value problem ($z\in[0, H]$)

$$\begin{cases} \hat{K}\Phi_n = 0, \ \ \Phi_n|_{\Gamma_0} = 0, \ \ \Phi_n|_{\Gamma_h^-} = \hat{R}_1\,\Phi_{n-1} + \hat{T}_{21}\,\Phi_{n-1}, \\ \Phi_n|_{\Gamma_h^+} = \hat{R}_2\,\Phi_{n-1} + \hat{T}_{12}\,\Phi_{n-1}, \ \ \Phi_n|_{\Gamma_H} = 0 \end{cases} \tag{27}$$

is separated for the sources into two problems: for the layer $z\in[0, h]$

$$\{\hat{K}\,\Phi_{{\rm a},n} = 0, \ \Phi_{{\rm a},n}|_{\Gamma_0} = 0, \quad \Phi_{{\rm a},n}|_{\Gamma_h^-} = \hat{R}_1\,\Phi_{{\rm a},n-1} + \hat{T}_{21}\,\Phi_{{\rm oc},n-1} \tag{28}$$

and for the layer $z\in[h, H]$

$$\{\hat{K}\,\Phi_{{\rm oc},n} = 0, \quad \Phi_{{\rm oc},n}|_{\Gamma_H} = 0, \quad \Phi_{{\rm oc},n}|_{\Gamma_h^+} = \hat{R}_2\,\Phi_{{\rm oc},n-1} + \hat{T}_{12}\,\Phi_{{\rm a},n-1}. \tag{29}$$

For the vector function $\mathbf{f} = \{f_{\rm a}(s_*; h, r_\perp, s), f_{\rm oc}(s_*; h, r_\perp, s)\}$ with the parameters s_*, let us define the linear functional

$$(\boldsymbol{\Theta}, \mathbf{f}) = \{(\Theta_{\rm a}, f_{\rm a}), (\Theta_{\rm oc}, f_{\rm oc})\}, \tag{30}$$

with the components being the linear functionals ($s\in\Omega$):

$$[(\Theta_{\rm a}, f_{\rm a})](s_*; z, r_\perp, s) \equiv \equiv \frac{1}{2\pi}\int\limits_{\Omega^-} {\rm d}s^- \int\limits_{-\infty}^{\infty} \Theta_{\rm a}(s^-; z, r_\perp - r'_\perp, s)\,f_{\rm a}(s_*; h, r'_\perp, s^-)\,{\rm d}r'_\perp, \quad z \in [0, h], \tag{31}$$

$$[(\Theta_{\rm oc}, f_{\rm oc})](s_*; z, r_\perp, s) \equiv \equiv \frac{1}{2\pi}\int\limits_{\Omega^+} {\rm d}s^+ \int\limits_{-\infty}^{\infty} \Theta_{\rm oc}(s^+; z, r_\perp - r'_\perp, s)\,f_{\rm oc}(s_*; h, r'_\perp, s^+)\,{\rm d}r'_\perp, \quad z \in [h, H]. \tag{32}$$

The parameters s_* may be absent in the functions $f_{\rm a}$ and $f_{\rm oc}$.

The interaction of radiation with the interface is described by the vector functional, with the kernels being the influence functions of the atmosphere and the ocean:

$$[\hat{M}\,\mathbf{f}](h, r_\perp, s) \equiv \hat{P}(\boldsymbol{\Theta}, \mathbf{f}) = \begin{bmatrix} \hat{R}_1(\Theta_{\rm a}, f_{\rm a}) + \hat{T}_{21}(\Theta_{\rm oc}, f_{\rm oc}) \\ \hat{T}_{12}(\Theta_{\rm a}, f_{\rm a}) + \hat{R}_2(\Theta_{\rm oc}, f_{\rm oc}) \end{bmatrix}, \tag{33}$$

where the matrix $\hat{P}$ is

$$\hat{P} \equiv \begin{bmatrix} \hat{R}_1 & \hat{T}_{21} \\ \hat{T}_{12} & \hat{R}_2 \end{bmatrix}.$$

Let us write down the explicit expressions for the components of functional (33):

$$[\hat{R}_1(\Theta_a, f_a)](s_*; h, r_\perp, s) =$$

$$= \int_{\Omega^+} [(\Theta_a, f_a)](s_*; h, r_\perp, s^+)\, q_1(r_\perp, s, s^+)\, ds^+ =$$

$$= \int_{\Omega^+} q_1(r_\perp, s, s^+)\, ds^+ \frac{1}{2\pi} \int_{\Omega^-} ds^- \int_{-\infty}^{\infty} f_a(s_*; h, r'_\perp, s^-) \times$$

$$\times\, \Theta_a(s^-; h, r_\perp - r'_\perp, s^+)\, dr'_\perp, \quad s \in \Omega^-; \qquad (34)$$

$$[\hat{R}_2(\Theta_{oc}, f_{oc})]\, (s_*; h, r_\perp, s) =$$

$$= \int_{\Omega^-} [(\Theta_{oc}, f_{oc})](s_*; h, r_\perp, s^-)\, q_2(r_\perp, s, s^-)\, ds^- =$$

$$= \int_{\Omega^-} q_2(r_\perp, s, s^-)\, ds^- \frac{1}{2\pi} \int_{\Omega^+} ds^+ \int_{-\infty}^{\infty} f_{oc}(s_*; h, r'_\perp, s^+) \times$$

$$\times\, \Theta_{oc}(s^+; h, r_\perp - r'_\perp, s^-)\, dr'_\perp, \quad s \in \Omega^+, \qquad (35)$$

$$[\hat{T}_{12}(\Theta_a, f_a)](s_*; h, r_\perp, s) =$$

$$= \int_{\Omega^+} [(\Theta_a, f_a)](s_*; h, r_\perp, s^+)\, t_{12}(r_\perp, s, s^+)\, ds^+ =$$

$$= \int_{\Omega^+} t_{12}(r_\perp, s, s^+)\, ds^+ \frac{1}{2\pi} \int_{\Omega^-} ds^- \int_{-\infty}^{\infty} f_a(s_*; h, r'_\perp, s^-) \times$$

$$\times\, \Theta_a(s^-; h, r_\perp - r'_\perp, s^+)\, dr'_\perp, \quad s \in \Omega^+; \qquad (36)$$

$$[\hat{T}_{21}(\Theta_{oc}, f_{oc})](s_*; h, r_\perp, s) =$$

$$= \int_{\Omega^-} [(\Theta_{oc}, f_{oc})](s_*; h, r_\perp, s^-)\, t_{21}(r_\perp, s, s^-)\, ds^- =$$

$$= \int_{\Omega^-} t_{21}\, (r_\perp, s, s^-)\, ds^- \frac{1}{2\pi} \int_{\Omega^+} ds^+ \int_{-\infty}^{\infty} f_{oc}(s_*; h, r'_\perp, s^+) \times$$

$$\times\, \Theta_{oc}(s^+; h, r_\perp - r'_\perp, s^-)\, dr'_\perp, \quad s \in \Omega^-. \qquad (37)$$

For $n = 2$, the solution to boundary-value problem (27) takes the form of two linear functionals for two components being the solutions of boundary-value problems (28) and (29)

$$\Phi_{a,2} = (\Theta_a, \hat{R}_1\, \Phi_{a,1} + \hat{T}_{21}\, \Phi_{oc,1}) =$$
$$= (\Theta_a, \hat{R}_1\, \Phi_{a,1}) + (\Theta_a, \hat{T}_{21}\, \Phi_{oc,1}),$$

$$\Phi_{oc,2} = (\Theta_{oc}, \hat{R}_2\, \Phi_{oc,1} + \hat{T}_{12}\, \Phi_{a,1}) =$$
$$= (\Theta_{oc}, \hat{R}_2\, \Phi_{oc,1}) + (\Theta_{oc}, \hat{T}_{12}\, \Phi_{a,1}).$$

Using representations (24) and (25), functionals (31) and (32), and operator definitions (1)–(4), we derive

$$[\hat{R}_1\, \Phi_{a,1}](h, r_\perp, s) = \int_{\Omega^+} q_1(r_\perp, s, s_1^+) ds_1^+ \times$$

$$\times \frac{1}{2\pi} \int_{\Omega^-} ds_1^- \int_{-\infty}^{\infty} \Theta_a(s_1^-; z, r_\perp - r_{\perp 1}, s_1^+)\, E_a(r_{\perp 1}, s_1^-)\, dr_{\perp 1} =$$

$$= \frac{1}{2\pi} \int_{\Omega^-} ds_1^- \int_{-\infty}^{\infty} E_a(r_{\perp 1}, s_1^-)\, dr_{\perp 1} \times$$

$$\times \int_{\Omega^+} q_1(r_\perp, s, s_1^+)\, \Theta_a(s_1^-; z, r_\perp - r_{\perp 1}, s_1^+)\, ds_1^+ = \hat{R}_1(\Theta_a, E_a),$$

$$[\hat{T}_{21}\, \Phi_{oc,1}](h, r_\perp, s) = \int_{\Omega^-} t_{21}(r_\perp, s, s_1^-)\, ds_1^- \times$$

$$\times \frac{1}{2\pi} \int_{\Omega^+} ds_1^+ \int_{-\infty}^{\infty} \Theta_{oc}(s_1^+; z, r_\perp - r_{\perp 1}, s_1^-)\, E_{oc}(r_{\perp 1}, s_1^+)\, dr_{\perp 1} =$$

$$= \frac{1}{2\pi} \int_{\Omega^+} ds_1^+ \int_{-\infty}^{\infty} E_{oc}(r_\perp, s_1^+)\, dr_{\perp 1} \times$$

$$\times \int_{\Omega^-} t_{21}(r_\perp, s, s_1^-)\, \Theta_{oc}(s_1^+; z, r_\perp - r_{\perp 1}, s_1^-)\, ds_1^- =$$
$$= \hat{T}_{21}(\Theta_{oc}, E_{oc}),$$

$$[\hat{R}_2\, \Phi_{oc,1}](h, r_\perp, s) = \int_{\Omega^-} q_2(r_\perp, s, s_1^-)\, ds_1^- \times$$

$$\times \frac{1}{2\pi} \int_{\Omega^+} ds_1^+ \int_{-\infty}^{\infty} E_{oc}(r_{\perp 1}, s_1^+)\, \Theta_{oc}(s_1^+; z, r_\perp - r_{\perp 1}, s_1^-)\, dr_{\perp 1} =$$

$$= \frac{1}{2\pi} \int_{\Omega^+} ds_1^+ \int_{-\infty}^{\infty} E_{oc}(r_{\perp 1}, s_1^+)\, dr_{\perp 1} \times$$

$$\times \int_{\Omega^-} q_2(r_\perp, s, s_1^-)\, \Theta_{oc}(s_1^+; z, r_\perp - r_{\perp 1}, s_1^-)\, ds_1^- =$$
$$= \hat{R}_2(\Theta_{oc}, E_{oc}),$$

$$[\hat{T}_{12}\, \Phi_{a,1}](h, r_\perp, s) = \int_{\Omega^+} t_{12}(r_\perp, s, s_1^+)\, ds_1^+ \times$$

$$\times \frac{1}{2\pi} \int_{\Omega^-} ds_1^- \int_{-\infty}^{\infty} E_a(r_{\perp 1}, s_1^-)\, \Theta_a(s_1^-; z, r_\perp - r_{\perp 1}, s_1^+)\, dr_{\perp 1} =$$

$$= \frac{1}{2\pi} \int_{\Omega^-} ds_1^- \int_{-\infty}^{\infty} E_a(r_{\perp 1}, s_1^-)\, dr_{\perp 1} \times$$

$$\times \int_{\Omega^+} t_{12}(r_\perp, s, s_1^+)\, \Theta_a(s_1^-; z, r_\perp - r_{\perp 1}, s_1^+)\, ds_1^+ = \hat{T}_{12}(\Theta_a, E_a).$$

As is seen, the isoplanarity property is not fulfilled; therefore,

$$\hat{R}_1(\Theta_a, E_a) \neq (\hat{R}_1 \Theta_a, E_a),\quad \hat{R}_2(\Theta_{oc}, E_{oc}) \neq (\hat{R}_2 \Theta_{oc}, E_{oc}),$$

$$\hat{T}_{21}(\Theta_{oc}, E_{oc}) \neq (\hat{T}_{21} \Theta_{oc}, E_{oc}),\quad \hat{T}_{12}(\Theta_a, E_a) \neq (\hat{T}_{12} \Theta_a, E_a).$$

The second–order approximation of perturbation series (17) can be found in the explicit form as ($s \in \Omega$)

$$\Phi_{a,2}(z, r_\perp, s) = \frac{1}{2\pi} \int_{\Omega^-} ds_2^- \int_{-\infty}^{\infty} \Theta_a(s_2^-; z, r_\perp - r_{\perp 2}, s) \times$$
$$\times [\hat{R}_1 \Phi_{a,1}](h, r_{\perp 2}, s_2^-)\, dr_{\perp 2} +$$
$$+ \frac{1}{2\pi} \int_{\Omega^-} ds_2^- \int_{-\infty}^{\infty} \Theta_a(s_2^-; z, r_\perp - r_{\perp 2}, s) \times$$
$$\times [\hat{T}_{21} \Phi_{oc,1}]\, (h, r_{\perp 2}, s_2^-)\, dr_{\perp 2} =$$
$$= \frac{1}{2\pi} \int_{\Omega^-} ds_2^- \int_{-\infty}^{\infty} \Theta_a(s_2^-; z, r_\perp - r_{\perp 2}, s) \times$$
$$\times \{[\hat{R}_1 (\Theta_a, E_a)](h, r_{\perp 2}, s_2^-) +$$
$$+ [\hat{T}_{21}(\Theta_{oc}, E_{oc})]\, (h, r_{\perp 2}, s_2^-)\}\, dr_{\perp 2} = (\Theta_a, \hat{R}_1(\Theta_a, E_a)) +$$
$$+ (\Theta_a, \hat{T}_{21}(\Theta_{oc}, E_{oc})),\quad z \in [0, h],$$

$$\Phi_{oc,2}(z, r_\perp, s) = \frac{1}{2\pi} \int_{\Omega^+} ds_2^+ \int_{-\infty}^{\infty} \Theta_{oc}(s_2^+; z, r_\perp - r_{\perp 2}, s) \times$$
$$\times [\hat{R}_2 \Phi_{oc,1}]\, (h, r_{\perp 2}, s_2^+)\, dr_{\perp 2} +$$
$$+ \frac{1}{2\pi} \int_{\Omega^+} ds_2^+ \int_{-\infty}^{\infty} \Theta_{oc}(s_2^+; z, r_\perp - r_{\perp 2}, s) \times$$
$$\times [\hat{T}_{12} \Phi_{a,1}](h, r_{\perp 2}, s_2^+)\, dr_{\perp 2} =$$
$$= \frac{1}{2\pi} \int_{\Omega^+} ds_2^+ \int_{-\infty}^{\infty} \Theta_{oc}(s_2^+; z, r_\perp - r_{\perp 2}, s) \times$$
$$\times \{[\hat{R}_2(\Theta_{oc}, E_{oc})](h, r_{\perp 2}, s_2^+) +$$
$$+ [\hat{T}_{12}(\Theta_a, E_a)](h, r_{\perp 2}, s_2^+)\}\, dr_{\perp 2} = (\Theta_{oc}, \hat{R}_2(\Theta_{oc}, E_{oc})) +$$
$$+ (\Theta_{oc}, \hat{T}_{12}(\Theta_a, E_a)),\quad z \in [h, H].$$

We turn our attention to the following fact: at the interface level $z = h$ the components $\Phi_{a,n}$ and $\Phi_{oc,n}$ are defined for the entire sphere of directions $s \in \Omega$.

Now let us write the first three approximations in the vector operator form and use definition (33):

$$\mathbf{\Phi}_1 = \begin{bmatrix} \Phi_{a,1} \\ \Phi_{oc,1} \end{bmatrix} = \begin{bmatrix} (\Theta_a, E_a) \\ (\Theta_{oc}, E_{oc}) \end{bmatrix} = (\mathbf{\Theta}, \mathbf{E}), \tag{38}$$

$$\mathbf{F}_1 = \hat{P}\, \mathbf{\Phi}_1 = \hat{P}(\mathbf{\Theta}, \mathbf{E}) = \hat{M}\, \mathbf{E},$$

$$\mathbf{\Phi}_2 = (\mathbf{\Theta}, \mathbf{F}_1) = (\mathbf{\Theta}, \hat{M}\, \mathbf{E}) = (\mathbf{\Theta}, \hat{P}\, \mathbf{\Phi}_1) = (\mathbf{\Theta}, \hat{P}(\mathbf{\Theta}, \mathbf{E})),$$

$$\mathbf{F}_2 = \hat{P}\, \mathbf{\Phi}_2 = \hat{P}(\mathbf{\Theta}, \mathbf{F}_1) = \hat{M}\, \mathbf{F}_1 = \hat{M}^2\, \mathbf{E},$$

$$\mathbf{\Phi}_3 = (\mathbf{\Theta}, \mathbf{F}_2) = (\mathbf{\Theta}, \hat{P}\, \mathbf{F}_2) = (\mathbf{\Theta}, \hat{M}\, \mathbf{F}_1) = (\mathbf{\Theta}, \hat{M}^2\, \mathbf{E}).$$

It can be shown that two successive $(n-1)$th and nth approximations are related by the recursion relation

$$\mathbf{\Phi}_n = (\mathbf{\Theta}, \hat{P}\, \mathbf{\Phi}_{n-1}) = (\mathbf{\Theta}, \hat{M}^{n-1}\, \mathbf{E}), \tag{39}$$

which includes matrix operator describing a single event of radiation passage through the interface at the level $z = h$ with regard for multiple scattering in both media. As a result,

$$\mathbf{\Phi} = \sum_{n=1}^{\infty} \mathbf{\Phi}_n = (\mathbf{\Theta}, \mathbf{E}) + \left(\mathbf{\Theta}, \sum_{n=1}^{\infty} \hat{M}^n\, \mathbf{E}\right) = (\mathbf{\Theta}, \hat{Z}\, \mathbf{E}),$$

$$\hat{Z}\, \mathbf{E} \equiv \sum_{n=0}^{\infty} \hat{M}^n\, \mathbf{E} \tag{40}$$

is the sum of the Neumann series over the multiplicity of radiation passage through the interface. The solution of boundary-value problem (16) represented in the form of functional

$$\mathbf{\Phi} = (\mathbf{\Theta}, \hat{Z}\, \mathbf{E}) \tag{41}$$

is the optical transfer operator of the atmosphere–ocean system. It establishes the explicit relation between the radiation measured and scenario (40) at the interface. The scenario (40), in its turn, is expressed explicitly in terms of the reflection and transmission characteristics of the interface at its given illumination with the help of the influence functions of the atmosphere and the ocean.

EQUATIONS FOR THE SFC OF THE ATMOSPHERE AND OCEAN AND THE OPTICAL TRANSFER OPERATOR

With the help of the Fourier transform over the coordinate $r_\perp = (x, y)$

$$\check{f}(p) \equiv F[f(r_\perp)](p) = \int_{-\infty}^{\infty} f(r_\perp) \exp[i(p, r_\perp)]\, dr_\perp =$$

$$= \int\int_{-\infty}^{\infty} f(x, y) \exp[i\,(p_x\, x + p_y\, y)]\, dx\, dy,$$

where the spatial frequency $p = (p_x, p_y)$ takes only real values ($-\infty < p_x, p_y < \infty$), the boundary-value problem

$$\begin{cases} \hat{K}\Phi = F_b,\quad \Phi|_{\Gamma_0} = f_0,\quad \Phi|_{\Gamma_H} = f_H, \\ \Phi|_{\Gamma_h^+} = f_h^+,\quad \Phi|_{\Gamma_h^-} = f_h^- \end{cases}$$

is reduced to the 1D parametric boundary-value problem[2]

$$\begin{cases} \hat{L}(p)\,\check{\Phi} = \check{F}_b, \quad \check{\Phi}|_{\Gamma_0} = \check{f}_0, \quad \check{\Phi}|_{\Gamma_H} = \check{f}_H, \\ \check{\Phi}|_{\Gamma_h^+} = \check{f}_h^+, \quad \check{\Phi}|_{\Gamma_h^-} = \check{f}_h^- \end{cases} \tag{42}$$

with the operator

$$\hat{L}(p) \equiv \hat{D}_z - i(p, s_\perp) - \hat{S},$$
$$(p, s_\perp) = p_x \sin\vartheta \cos\varphi + p_y \sin\vartheta \sin\varphi \,.$$

The Fourier transforms are indicated by the symbol "∨" atop. The boundary-value problem (42) differs from the standard 1D boundary-value problem[2] by the presence of the anisotropic complex total extinction coefficient

$$\check{\sigma}_t(z, p, s_\perp) = \sigma_t(z) - i(p, s_\perp),$$

which is p-dependent. If the solar flux ($\check{F}_b = \check{f}_0 = \check{f}_H = 0$) is the source of radiation, then boundary-value problem (42) is separated into two problems: for the layer $z\in[0,h]$

$$\{\hat{L}(p)\,\check{\Phi} = 0, \quad \check{\Phi}|_{\Gamma_0} = 0, \quad \check{\Phi}|_{\Gamma_h^-} = \check{f}_h^- \tag{43}$$

and for the layer $z\in[h, H]$

$$\{\hat{L}(p)\,\check{\Phi} = 0, \quad \check{\Phi}|_{\Gamma_H} = 0, \quad \check{\Phi}|_{\Gamma_h^+} = \check{f}_h^+. \tag{44}$$

As was shown in Refs. 1–8, the solutions to boundary-value problems (9)–(11) in the form of Fourier transforms of the light field components $\check{\Phi}_{aR}^0$, $\check{\Phi}_{aR}^d$, and $\check{\Phi}_{aR}$, as well as the solution to boundary-value problem (43), are determined via the spatial frequency characteristic of the atmosphere $\Psi_a(s^-; z, p, s)$ being the solution of the boundary-value problem for the layer $z \in [0, h]$

$$\{\hat{L}(p)\,\Psi_a = 0, \quad \Psi_a|_{\Gamma_0} = 0, \quad \Psi_a|_{\Gamma_h^-} = \check{f}_\delta(s^-; p, s), \tag{45}$$
$$\check{f}_\delta(s^-; p, s) = F[f_\delta(s^-; r_\perp, s)] = \delta(s - s^-).$$

The spatial-frequency characteristic and the influence function of the atmosphere are related by the direct and inverse Fourier transforms over the coordinate $r_\perp$:

$$\Psi_a(s^-; z, p, s) \equiv F[\Theta_a(s^-; z, r_\perp, s)] =$$
$$= \int_{-\infty}^{\infty} \Theta_a(s^-; z, r_\perp, s) \exp[i(p, r_\perp)]\, dr_\perp,$$

$$\Theta_a(s^-; z, r_\perp, s) = F^{-1}[\Psi_a(s^-; z, p, s)] =$$
$$= \frac{1}{(2\pi)^2} \int_{-\infty}^{\infty} \Psi_a(s^-; z, p, s) \exp[-i(p, r_\perp)]\, dp.$$

We will seek for the solution of boundary-value problem (16) in terms of the Fourier transforms. To do this, we introduce the perturbation series

$$\check{\Phi}_{oc}(z, p, s) = \sum_{n=1}^{\infty} \eta^n \check{\Phi}_n(z, p, s) \tag{46}$$

and two-component vectors

$$\check{\mathbf{\Phi}}_n = \{\check{\Phi}_{a,n}, \check{\Phi}_{oc,n}\}, \; \check{\mathbf{E}}_n = \{\check{E}_a, \check{E}_{oc}\}.$$

Now we write down the Fourier transforms for reflection and transmission operators (1)–(4):

$$[\check{R}_1\check{\Phi}](h, p, s) \equiv F[\hat{R}_1\,\Phi](h, p, s) =$$
$$= \frac{1}{(2\pi)^2} \int_{-\infty}^{\infty} dp' \int_{\Omega^+} \check{q}_1(p-p', s, s^+)\, \check{\Phi}(h, p', s^+)\, ds^+ =$$
$$= \frac{1}{(2\pi)^2} \int_{-\infty}^{\infty} dp' \int_{\Omega^+} \check{q}_1(p', s, s^+)\, \check{\Phi}(h, p - p', s^+)\, ds^+ ,$$

$$[\check{R}_2\check{\Phi}](h, p, s) \equiv F[\hat{R}_2\,\Phi](h, p, s) =$$
$$= \frac{1}{(2\pi)^2} \int_{-\infty}^{\infty} dp' \int_{\Omega^-} \check{q}_2(p - p', s, s^-)\, \check{\Phi}(h, p', s^-)\, ds^- =$$
$$= \frac{1}{(2\pi)^2} \int_{-\infty}^{\infty} dp' \int_{\Omega^-} \check{q}_2(p', s, s^-)\, \check{\Phi}(h, p - p', s^-)\, ds^- ,$$

$$[\check{T}_{12}\,\check{\Phi}](h, p, s) \equiv F[\hat{T}_{12}\,\Phi](h, p, s) =$$
$$= \frac{1}{(2\pi)^2} \int_{-\infty}^{\infty} dp' \int_{\Omega^+} \check{\Phi}(h, p', s^+)\, \check{t}_{12}(p - p', s, s^+)\, ds^+ =$$
$$= \frac{1}{(2\pi)^2} \int_{-\infty}^{\infty} dp' \int_{\Omega^+} \check{\Phi}(h, p - p', s^+)\, \check{t}_{12}(p', s, s^+)\, ds^+ ,$$

$$[\check{T}_{21}\,\check{\Phi}](h, p, s) \equiv F[\hat{T}_{21}\,\Phi](h, p, s) =$$
$$= \frac{1}{(2\pi)^2} \int_{-\infty}^{\infty} dp' \int_{\Omega^-} \check{\Phi}(h, p', s^-)\, \check{t}_{21}(p - p', s, s^-)\, ds^- =$$
$$= \frac{1}{(2\pi)^2} \int_{-\infty}^{\infty} dp' \int_{\Omega^-} \check{\Phi}(h, p - p', s^-)\, \check{t}_{21}(p', s, s^-)\, ds^- .$$

In the linear approximation (n=1) boundary-value problem (19) in terms of the Fourier transforms is separated into two problems: for the ocean ($z\in[h, H]$)

$$\{\hat{L}(p)\,\check{\Phi}_{oc,1} = 0, \quad \check{\Phi}_{oc,1}|_{\Gamma_H} = 0, \quad \check{\Phi}_{oc,1}|_{\Gamma_h^+} = \check{E}_{oc} \tag{47}$$

and for the atmosphere ($z\in[0, h]$)

$$\{\hat{L}(p)\,\check{\Phi}_{\mathrm{a},1} = 0, \quad \check{\Phi}_{\mathrm{a},1}|_{\Gamma_0} = 0, \quad \check{\Phi}_{\mathrm{a},1}|_{\Gamma_h^-} = \check{E}_{\mathrm{a}}. \qquad (48)$$

Let us now find the Fourier transforms of functionals (22) and (23)

$$\check{E}_{\mathrm{a}}(p, s) = \frac{1}{2\pi}\int_{\Omega^-} \delta(s - s^-)\,\check{E}_{\mathrm{a}}(p, s^-)\,\mathrm{d}s^-,$$

$$\check{E}_{\mathrm{oc}}(p, s) = \frac{1}{2\pi}\int_{\Omega^+} \delta(s - s^+)\,\check{E}_{\mathrm{oc}}(p, s^+)\,\mathrm{d}s^+,$$

then the solutions of problems (47) and (48) can be written in the form of linear functionals ($s \in \Omega$)

$$\check{\Phi}_{\mathrm{a},1}(z, p, s) = (\Psi_{\mathrm{a}}, \check{E}_{\mathrm{a}}) = \frac{1}{2\pi}\int_{\Omega^-} \Psi_{\mathrm{a}}(s_1^-; z, p, s)\,\check{E}_{\mathrm{a}}(p, s_1^-)\,\mathrm{d}s_1^-, \quad z \in [0, h],$$

$$\check{\Phi}_{\mathrm{oc},1}(z, p, s) = (\Psi_{\mathrm{oc}}, \check{E}_{\mathrm{oc}}) = \frac{1}{2\pi}\int_{\Omega^+} \Psi_{\mathrm{oc}}(s_1^+; z, p, s)\,\check{E}_{\mathrm{oc}}(p, s_1^+)\,\mathrm{d}s_1^+, \quad z \in [h, H],$$

whose kernels are the SFC of the atmosphere $\Psi_{\mathrm{a}}(s^-; z, p, s)$, being the solution of boundary-value problem (45), and the SFC of the ocean

$$\Psi_{\mathrm{oc}}(s^+; z, p, s) \equiv F[\Theta_{\mathrm{oc}}(s^+; z, r_\perp, s)] = \int_{-\infty}^{\infty} \Theta_{\mathrm{oc}}(s^+; z, r_\perp, s)\exp[i(p, r_\perp)]\,\mathrm{d}r_\perp,$$

$$\Theta_{\mathrm{oc}}(s^+; z, r_\perp, s) = F^{-1}[\Psi_{\mathrm{oc}}(s^+; z, p, s)] = \frac{1}{(2\pi)^2}\int_{-\infty}^{\infty} \Psi_{\mathrm{oc}}(s^+; z, p, s)\exp[-\,i(p, r_\perp)]\,\mathrm{d}p,$$

which is the solution of the boundary-value problem for the layer $z \in [h, H]$

$$\{\hat{L}(p)\,\Psi_{\mathrm{oc}} = 0, \quad \Psi_{\mathrm{oc}}|_{\Gamma_H} = 0, \quad \Psi_{\mathrm{oc}}|_{\Gamma_h^+} = \check{f}_\delta(s^+; p, s) \qquad (49)$$

with the source

$$\check{f}_\delta(s^+; p, s) = F[f_\delta(s^+; r_\perp, s)] = \delta(s - s^+).$$

The Fourier transform of the influence vector-function is the SFC vector

$$\mathbf{\Psi} \equiv F[\mathbf{\Theta}] = \{\Psi_{\mathrm{a}}, \Psi_{\mathrm{oc}}\},$$

whose components are the SFCs of the atmosphere $\Psi_{\mathrm{a}}(s^-; z, p, s)$ and the ocean $\Psi_{\mathrm{oc}}(s^+; z, p, s)$.

For the vector-function $\check{\mathbf{f}} = \{\check{f}_{\mathrm{a}}(s_*; h, p, s), \check{f}_{\mathrm{oc}}(s_*; h, p, s)\}$ with the parameters s_* let us define the vector linear functional ($s \in \Omega$):

$$(\mathbf{\Psi}, \check{\mathbf{f}}) = \{(\Psi_{\mathrm{a}}, \check{f}_{\mathrm{a}}), (\Psi_{\mathrm{oc}}, \check{f}_{\mathrm{oc}})\},$$

with the components being the Fourier transforms of linear functionals (31) and (32):

$$[(\mathbf{\Psi}_{\mathrm{a}}, \check{f}_{\mathrm{a}})](s_*; z, p, s) = F[(\Theta_{\mathrm{a}}, f_{\mathrm{a}})] = \frac{1}{2\pi}\int_{\Omega^-} \Psi_{\mathrm{a}}(s^-; z, p, s)\,\check{f}_{\mathrm{a}}(s_*; h, p, s^-)\,\mathrm{d}s^-, \; z \in [0, h], \qquad (50)$$

$$[(\mathbf{\Psi}_{\mathrm{oc}}, \check{f}_{\mathrm{oc}})](s_*; z, p, s) = F[(\Theta_{\mathrm{oc}}, f_{\mathrm{oc}})] = \frac{1}{2\pi}\int_{\Omega^+} \Psi_{\mathrm{oc}}(s^+; z, p, s)\,\check{f}_{\mathrm{oc}}(s_*; h, p, s^+)\,\mathrm{d}s^+, \; z \in [h, H]. \qquad (51)$$

The interaction of radiation with the interface is described in terms of the Fourier transforms by the functional whose kernels are the SFCs of the atmosphere and the ocean

$$[\hat{Q}\,\check{\mathbf{f}}](h, p, s) \equiv F[\hat{P}\,\mathbf{f}] = \hat{G}(\mathbf{\Psi}, \check{\mathbf{f}}) = \begin{bmatrix} \check{R}_1(\Psi_{\mathrm{a}}, \check{f}_{\mathrm{a}}) + \check{T}_{21}(\Psi_{\mathrm{oc}}, \check{f}_{\mathrm{oc}}) \\ \check{T}_{12}(\Psi_{\mathrm{a}}, \check{f}_{\mathrm{a}}) + \check{R}_2(\Psi_{\mathrm{oc}}, \check{f}_{\mathrm{oc}}) \end{bmatrix}, \qquad (52)$$

where the operator matrix is

$$\hat{G} \equiv \begin{bmatrix} \check{R}_1 & \check{T}_{21} \\ \check{T}_{12} & \check{R}_2 \end{bmatrix}.$$

Let us now take the Fourier transform of Eqs. (34)–(37) and find the expressions for the components of functional (52) in the explicit form

$$F[\hat{R}_1(\Theta_{\mathrm{a}}, f_{\mathrm{a}})](s_*; h, p, s) = \frac{1}{2\pi}\int_{\Omega^-}\mathrm{d}s^-\int_{\Omega^+}\mathrm{d}s^+\left\{\frac{1}{(2\pi)^2}\int_{-\infty}^{\infty}\check{q}_1(p - p', s, s^+)\,\check{f}_{\mathrm{a}}(s_*; h, p', s^-)\,\Psi_{\mathrm{a}}(s^-; z, p', s^+)\,\mathrm{d}p'\right\} =$$

$$= \frac{1}{2\pi}\int_{\Omega^-}\mathrm{d}s^-\,\frac{1}{(2\pi)^2}\int_{-\infty}^{\infty}\check{f}_{\mathrm{a}}(s_*; h, p', s^-)\,\mathrm{d}p'\int_{\Omega^+}\Psi_{\mathrm{a}}(s^-; z, p', s^+)\,\check{q}_1(p - p', s, s^+)\,\mathrm{d}s^+ =$$

$$= \frac{1}{(2\pi)^2}\int_{-\infty}^{\infty}\mathrm{d}p'\int_{\Omega^+}\check{q}_1(p - p', s, s^+)\,[(\Psi_{\mathrm{a}}, \check{f}_{\mathrm{a}})](s_*; h, p', s^+)\,\mathrm{d}s^+ = [\check{R}_1(\Psi_{\mathrm{a}}, \check{f}_{\mathrm{a}})](s_*; h, p, s). \qquad (53)$$

Similarly we find

$$F[\hat{R}_2(\Theta_{\mathrm{oc}}, f_{\mathrm{oc}})](s_*; h, p, s) = \frac{1}{(2\pi)^2}\int\limits_{-\infty}^{\infty} \mathrm{d}p' \int\limits_{\Omega^-} \check{q}_2(p-p', s, s^-)\,[(\Psi_{\mathrm{oc}}, \check{f}_{\mathrm{oc}})](s_*; h, p', s^-)\mathrm{d}s^- = [\check{R}_2(\Psi_{\mathrm{oc}}, \check{f}_{\mathrm{oc}})](s_*; h, p, s), \quad (54)$$

$$F[\hat{T}_{12}(\Theta_{\mathrm{a}}, f_{\mathrm{a}})](s_*; h, p, s) = \frac{1}{(2\pi)^2}\int\limits_{-\infty}^{\infty} \mathrm{d}p' \int\limits_{\Omega^+} \check{t}_{12}(p-p', s, s^+)[(\Psi_{\mathrm{a}}, \check{f}_{\mathrm{a}})](s_*; h, p', s^+)\,\mathrm{d}s^+ = [\check{T}_{12}(\Psi_{\mathrm{a}}, \check{f}_{\mathrm{a}})](s_*; h, p, s), \quad (55)$$

$$F[\hat{T}_{21}(\Theta_{\mathrm{oc}}, f_{\mathrm{oc}})](s_*; h, p, s) = \frac{1}{(2\pi)^2}\int\limits_{-\infty}^{\infty} \mathrm{d}p' \int\limits_{\Omega^-} \check{t}_{21}(p-p', s, s^-)\,[(\Psi_{\mathrm{oc}}, \check{f}_{\mathrm{oc}})](s_*; h, p', s^-)\,\mathrm{d}s^- = [\check{T}_{21}(\Psi_{\mathrm{oc}}, \check{f}_{\mathrm{oc}})](s_*; h, p, s). \quad (56)$$

In Eqs. (53)–(56) the condition of isoplanarity is not fulfilled; therefore,

$$\check{R}_1(\Psi_{\mathrm{a}}, \check{f}_{\mathrm{a}}) \neq (\check{R}_1\,\Psi_{\mathrm{a}}, \check{f}_{\mathrm{a}}),\ \check{R}_2(\Psi_{\mathrm{oc}}, \check{f}_{\mathrm{oc}}) \neq (\check{R}_2\,\Psi_{\mathrm{oc}}, \check{f}_{\mathrm{oc}}),$$

$$\check{T}_{21}(\Psi_{\mathrm{oc}}, \check{f}_{\mathrm{oc}}) \neq (\check{T}_{21}\,\Psi_{\mathrm{oc}}, \check{f}_{\mathrm{oc}}),\ \check{T}_{12}(\Psi_{\mathrm{a}}, \check{f}_{\mathrm{a}}) \neq (\check{T}_{12}\,\Psi_{\mathrm{a}}, \check{f}_{\mathrm{a}}).$$

For $n = 2$ the solution to boundary-value problem (28) in terms of the Fourier transforms is determined via the SFC $\Psi_{\mathrm{a}}(s^-; z, p, s)$:

$$\check{\Phi}_{\mathrm{a},2}(z, p, s) = (\Psi_{\mathrm{a}}, F[\hat{R}_1\,\Phi_{\mathrm{a},1}]) + (\Psi_{\mathrm{a}}, F[\hat{T}_{21}\,\Phi_{\mathrm{oc},1}]),$$

whereas the solution of boundary-value problem (29) – via the SFC $\Psi_{\mathrm{oc}}(s^-; z, p, s)$:

$$\check{\Phi}_{\mathrm{oc},2}(z, p, s) = (\Psi_{\mathrm{oc}}, F[\hat{R}_2\,\Phi_{\mathrm{oc},1}]) + (\Psi_{\mathrm{oc}}, F[\hat{T}_{12}\,\Phi_{\mathrm{a},1}]).$$

In the second approximation, we derive the components in the explicit form:

$$\check{\Phi}_{\mathrm{a},2}(z, p, s) = \frac{1}{2\pi}\int\limits_{\Omega^-} \Psi_{\mathrm{a}}(s_2^-; z, p, s)\,\mathrm{d}s_2^- \times$$
$$\times \frac{1}{(2\pi)^2}\int\limits_{-\infty}^{\infty} \mathrm{d}p_1 \int\limits_{\Omega^+} \check{q}_1(p - p_1, s_2^-, s_1^+) \times$$
$$\times \left\{\frac{1}{2\pi}\int\limits_{\Omega^-} \Psi_{\mathrm{a}}(s_1^-; h, p_1, s_1^+)\,\check{E}_{\mathrm{a}}(p_1, s_1^-)\,\mathrm{d}s_1^-\right\} \mathrm{d}s_1^+ +$$
$$+ \frac{1}{2\pi}\int\limits_{\Omega^-} \Psi_{\mathrm{a}}(s_2^-; z, p, s)\,\mathrm{d}s_2^- \times$$
$$\times \frac{1}{(2\pi)^2}\int\limits_{-\infty}^{\infty} \mathrm{d}p_1 \int\limits_{\Omega^-} \check{t}_{21}(p - p_1, s_2^-, s_1^-) \times$$
$$\times \left\{\frac{1}{2\pi}\int\limits_{\Omega^+} \Psi_{\mathrm{oc}}(s_1^+; h, p_1, s_1^-)\,\check{E}_{\mathrm{oc}}(p_1, s_1^+)\,\mathrm{d}s_1^+\right\} \mathrm{d}s_1^- =$$
$$= (\Psi_{\mathrm{a}}, [\check{R}_1(\Psi_{\mathrm{a}}, \check{E}_{\mathrm{a}})]) + (\Psi_{\mathrm{oc}}, [\check{T}_{21}(\Psi_{\mathrm{oc}}, \check{E}_{\mathrm{oc}})]);$$

$$\check{\Phi}_{\mathrm{oc},2}(z, p, s) = \frac{1}{2\pi}\int\limits_{\Omega^+} \Psi_{\mathrm{oc}}(s_2^+; z, p, s)\,\mathrm{d}s_2^+ \times$$
$$\times \frac{1}{(2\pi)^2}\int\limits_{-\infty}^{\infty} \mathrm{d}p_1 \int\limits_{\Omega^-} \check{q}_2(p - p_1, s_2^+, s_1^-) \times$$
$$\times \left\{\frac{1}{2\pi}\int\limits_{\Omega^+} \Psi_{\mathrm{oc}}(s_1^+; h, p_1, s_1^-)\,\check{E}_{\mathrm{oc}}(p_1, s_1^+)\,\mathrm{d}s_1^+\right\} \mathrm{d}s_1^- +$$
$$+ \frac{1}{2\pi}\int\limits_{\Omega^+} \Psi_{\mathrm{oc}}(s_2^+; z, p, s)\,\mathrm{d}s_2^+ \times$$
$$\times \frac{1}{(2\pi)^2}\int\limits_{-\infty}^{\infty} \mathrm{d}p_1 \int\limits_{\Omega^+} \check{t}_{12}\,(p - p_1, s_2^+, s_1^+) \times$$
$$\times \left\{\frac{1}{2\pi}\int\limits_{\Omega^-} \Psi_{\mathrm{a}}(s_1^-; h, p_1, s_1^+)\,\check{E}_{\mathrm{a}}(p_1, s_1^-)\,\mathrm{d}s_1^-\right\} \mathrm{d}s_1^+ =$$
$$= (\Psi_{\mathrm{oc}}, [\check{R}_2(\Psi_{\mathrm{oc}}, \check{E}_{\mathrm{oc}})]) + (\Psi_{\mathrm{oc}}, [\check{T}_{12}(\Psi_{\mathrm{a}}, \check{E}_{\mathrm{a}})]).$$

Let us now write the first three approximations in the vector operator form using the definition of operation (52):

$$\check{\mathbf{\Phi}}_1 = \begin{bmatrix} \check{\Phi}_{\mathrm{a},1} \\ \check{\Phi}_{\mathrm{oc},1} \end{bmatrix} = \begin{bmatrix} (\Psi_{\mathrm{a}}, \check{E}_{\mathrm{a}}) \\ (\Psi_{\mathrm{oc}}, \check{E}_{\mathrm{oc}}) \end{bmatrix} = (\mathbf{\Psi}, \check{\mathbf{E}}),$$

$$\check{\mathbf{F}}_1 = \hat{G}\,\check{\mathbf{\Phi}}_1 = \hat{G}\,(\mathbf{\Psi}, \check{\mathbf{E}}) = \hat{Q}\,\check{\mathbf{E}}\,,$$

$$\check{\mathbf{\Phi}}_2 = (\mathbf{\Psi}, \check{\mathbf{F}}_1) = (\mathbf{\Psi}, \hat{Q}\,\check{\mathbf{E}}) = (\mathbf{\Psi}, \hat{G}\,\check{\mathbf{\Phi}}_1) = (\mathbf{\Psi}, \hat{G}(\mathbf{\Psi}, \check{\mathbf{E}})),$$

$$\mathbf{F}_2 = \hat{G}\,\check{\mathbf{\Phi}}_2 = \hat{G}\,(\mathbf{\Psi}, \check{\mathbf{F}}_1) = \hat{Q}\,\check{\mathbf{F}}_1 = \hat{Q}^2\,\check{\mathbf{E}}\,,$$

$$\check{\mathbf{\Phi}}_3 = (\mathbf{\Psi}, \check{\mathbf{F}}_2) = (\mathbf{\Psi}, \hat{G}\,\check{\mathbf{\Phi}}_2) = (\mathbf{\Psi}, \hat{Q}\,\check{\mathbf{F}}_1) = (\mathbf{\Psi}, \hat{Q}^2\,\check{\mathbf{E}}).$$

Two successive $(n-1)$th and nth approximations are related by the recurrence relation

$$\check{\mathbf{\Phi}}_n = (\mathbf{\Psi}, \hat{G}\,\check{\mathbf{\Phi}}_{n-1}) = (\mathbf{\Psi}, \hat{Q}^{n-1}\,\check{\mathbf{E}}),$$

in which the matrix operator describes a single event of the radiation interaction with the interface in terms of the Fourier transforms, and the multiple scattering in both media is taken into account. As a result,

$$\overset{\vee}{\mathbf{\Phi}} = \sum_{n=1}^{\infty} \overset{\vee}{\mathbf{\Phi}}_n = (\mathbf{\Psi}, \overset{\vee}{\mathbf{E}}) + \left(\mathbf{\Psi}, \sum_{n=1}^{\infty} \hat{Q}^n \overset{\vee}{\mathbf{E}}\right) = (\mathbf{\Psi}, \hat{Y} \overset{\vee}{\mathbf{E}}),$$

$$\hat{Y} \overset{\vee}{\mathbf{E}} = \sum_{n=0}^{\infty} \hat{Q}^n \overset{\vee}{\mathbf{E}} \qquad (57)$$

is the sum of the Neumann series (in terms of the Fourier transforms) over the multiplicity of radiation passage through the interface with regard for the multiple scattering in both media. The representation

$$\overset{\vee}{\mathbf{\Phi}} = (\mathbf{\Psi}, \hat{Y} \overset{\vee}{\mathbf{E}}) \qquad (58)$$

is the optical transfer operator of the atmosphere–ocean system, which establishes the explicit relation between the Fourier transform of the radiation to be measured and the Fourier transform of scenario (57) at the interface. In this case, expression (57) for the scenario expresses the explicit relation with the reflection and transmission characteristics of the interface at its given illumination with the help of the SFCs of the atmosphere and the ocean.

ACKNOWLEDGMENTS

This work was supported in part by Russian Fundamental Research Foundation Grant No. 93-05-08542.

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