ON THE RELATIONSHIP BETWEEN THE CHARACTERISTICS OF LIGHT SCATTERED BY A MACROSCOPIC PARTICLE AT DIFFERENT ANGLES

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General theorems of electrodynamics state the existence of the direct relations between the amplitudes of light scattered by a macroscopic particle at different angles. Any essential restrictions are not imposed on the properties of these particles. The example of large particles is considered as a specific illustration.

1. INTRODUCTION

In order to outline the problem, let us remind that the relationships between scattering at different angles can play an important role in many optical diagnostic methods. For example, in laser sensing of the atmosphere the relationship between scattering in the "forward" and "backward" directions (an extinction coefficient and a radar cross section) becomes the problem of principle.[1] We bear in mind the relations of the type

$$A(\mathbf{r}_0) = \sum_j b_j(\mathbf{r}_0, \mathbf{r}_{0j})\, A(r_{0j}) + a(\mathbf{r}_0) \quad (1)$$

where the quantities $A(\mathbf{r}_0)$ are related to the amplitude of light scattering in the direction of the unit vector $\mathbf{r}_0$. The sum is taken over the set of scattering angles (with the "weight" b_j) and $a(\mathbf{r}_0)$ is a certain known function.

Formulation of the problem on the light scattering by a single particle is presented in Section 2. The "theorem of existence" of the relations of the type (1) is proved in Section 3. Its description in detail which is suitable for the analysis of the large (compared with wavelength) crystal particles of arbitrary shape is given in Section 4. The particular cases of using the exact formulas are discussed in Section 5.

2. THE CANONICAL PROBLEM OF SCATTERING OF MONOCHROMATIC LIGHT (AT THE FREQUENCY ω) BY A SINGLE PARTICLE

Figure 1 shows the quantities used to solve the problem formulated in the title. The particle occupies the volume $V(\mathbf{r}' \in V)$. We search for the field $\mathbf{E}$ at the external point $\mathbf{r}$ and there

$$\mathbf{E} = \mathbf{E}_0 + \mathbf{E}^{(s)} . \quad (2)$$

The field of the source $\mathbf{E}_0$ is "vacuum"

$$\Delta\mathbf{E}_0 + \frac{\omega^2}{c^2}\mathbf{E}_0 = 0 , \ \mathrm{div}\mathbf{E}_0 = 0 , \ \mathbf{E}_0 = \mathbf{E}^{(0)}\exp\left[i\frac{\omega}{c}(\mathbf{k}_0\mathbf{r}) \right] \quad (3)$$

(with the constant value of $\mathbf{E}^{(0)}$ in the direction perpendicular to the unit vector $\mathbf{k}_0 = \mathbf{e}_3$ (see Fig. 1), c is the velocity of light), and $\mathbf{E}^{(s)}$ is interpreted as the scattered wave. The field $\tilde{\mathbf{E}}(\mathbf{r}')$ exists inside the particle. The tensor of the dielectric constant $\varepsilon_{\alpha\beta}$ is determined by particle material (certainly, $\varepsilon_{\alpha\beta}$ is the relative value in accordance with Eq. (3)), the medium being external for V is assumed to be vacuum, and the dipole moment of unit volume (induction $\mathbf{D} = \mathbf{E} + 4\pi\mathbf{P}$) is

$$\mathbf{P}(\mathbf{r}') = \sum_{\alpha,\,\beta=1}^{3} \mathbf{e}_\alpha \frac{\varepsilon_{\alpha\beta} - \delta_{\alpha\beta}}{4\pi} \mathbf{E}_\beta(\mathbf{r}') . \quad (4)$$

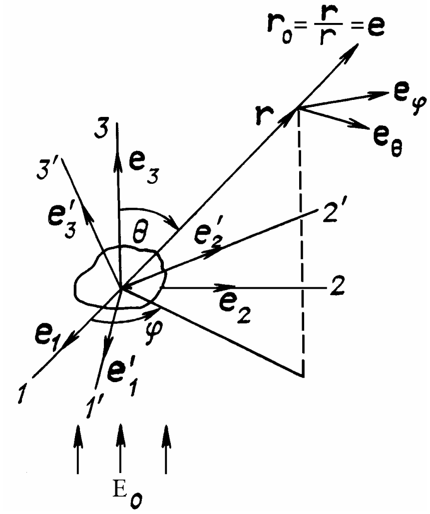

FIG. 1.

Figure 1 shows two Cartesian systems of coordinates (their origin is chosen somewhere inside V): the system without primes (with the unit vectors $\mathbf{e}_\alpha$) is affixed to incident wave (3) and the primed system (with the unit vectors $\mathbf{e}'_\alpha$) is coaxial with the principal axes of the tensor $\varepsilon_{\alpha\beta}$. Here, the spherical system of coordinates is introduced at the observation point $\mathbf{r}$. Finally, the last is assumed to be located in the far field for a particle:

$$\frac{\omega}{c} r \gg 1 , \ r \gg r' = |\mathbf{r}'| . \quad (5)$$

3. THE CONSEQUENCE OF SOME GENERAL THEOREMS OF ELECTRODYNAMICS

Let us first quote some rather general statements of electrodynamics which are referred to as "theorems"

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exceptionally for the convenience of references to them. As a result, we can write relations similar to Eq. (1).

Theorem 1. The system of Maxwell's differential equations is equivalent to the integral equation

$$\mathbf{E}(\mathbf{r}) = \mathbf{E}_0(\mathbf{r}) + \mathrm{rot\,rot}\int\limits_{(V)} \mathrm{d}r' G(\mathbf{r}|\mathbf{r}')P(\mathbf{r}') - \frac{4\pi}{3}\mathbf{P}(\mathbf{r}) , \quad (6)$$

where $G(\mathbf{r}|\mathbf{r}') = \exp(i\,\omega/c)|\mathbf{r}-\mathbf{r}'|/|\mathbf{r}-\mathbf{r}'|$ is the Green's function of the Helmholtz operator for the entire space.

In fact, Eq. (6) is presented in many manuals on electrodynamics, optics, and theory of scattering (for example, see Refs. 2–6).

The consequences of Theorem 1 are evident. For $\mathbf{r}' \in V$, as $\mathbf{E} \to \tilde{\mathbf{E}}$, Eqs. (6) and (4) are reduced to the integral equation for $\tilde{\mathbf{E}}$. After solving this equation, the formula for $\mathbf{E}^{(s)}$

$$\mathbf{E}^{(s)} = \mathrm{rot\,rot}\int\limits_{(V)} \mathrm{d}r' G(\mathbf{r}|\mathbf{r}')\mathbf{P}(\mathbf{r}') , \quad (7)$$

is derived from Eq. (2), since $\tilde{\mathbf{E}}$ in Eq. (4) will be known.

Theorem 2. In the far field for a particle

$$E_\theta^{(s)} = \frac{\omega^2}{c^2}\Pi_\theta \frac{\exp\left(i\frac{\omega}{c}r\right)}{r} \equiv A_\theta^{(s)}(\theta,\varphi)\frac{\exp\left(i\frac{\omega}{c}r\right)}{r} ,$$

$$E_\varphi^{(s)} = \frac{\omega^2}{c^2}\Pi_\varphi \frac{\exp\left(i\frac{\omega}{c}r\right)}{r} \equiv A_\varphi^{(s)}(\theta,\varphi)\frac{\exp\left(i\frac{\omega}{c}r\right)}{r} , \quad (8)$$

$$E_r^{(s)} = 0 , \; \mathbf{\Pi}(\mathbf{r}_0) = \int\limits_{(V)} \mathrm{d}\mathbf{r}' \exp\left(-i\frac{\omega}{c}\mathbf{r}_0\mathbf{r}'\right)\mathbf{P}(\mathbf{r}') .$$

Here, $A_\theta^{(s)}$ and $A_\varphi^{(s)}$ are called the amplitudes of the scattered field.

In fact, Eq. (8) is almost the obvious asymptote (5) for Eq. (7) (the details are given in Refs. 2, 7, and 8).

Theorem 3. The general structure of the field $\mathbf{E}^{(s)}$ from Eq. (2) is

$$\mathbf{E}^{(s)}(\mathbf{r}) = \int\limits_{(\mathbf{q}_0\mathbf{r}_0)\ge 0} \mathrm{d}\mathbf{q}_0 \sum_{\lambda=1}^{2} \mathrm{e}_\lambda(\mathbf{q}_0) B_\lambda(\mathbf{q}_0) \exp\left[i\frac{\omega}{c}(\mathbf{q}_0\mathbf{r})\right] ;$$

$$\mathrm{rot}\,\mathbf{E}^{(s)}(\mathbf{r}) = i\frac{\omega}{c}\int\limits_{(\mathbf{q}_0\mathbf{r}_0)\ge 0} \mathrm{d}\mathbf{q}_0 \sum_{\lambda=1}^{2} (\mathbf{q}_0\times\mathrm{e}_\lambda(\mathbf{q}_0)) \times$$

$$\times B_\lambda(\mathbf{q}_0)\exp\left[i\frac{\omega}{c}(\mathbf{q}_0\mathbf{r}_0)\right] ; \quad (9)$$

$$B_1(\mathbf{q}_0) = \frac{i\,\omega}{2\pi c}A_\theta^{(s)}(\mathbf{q}_0) , \qquad B_2(\mathbf{q}_0) = \frac{i\,\omega}{2\pi c}A_\varphi^{(s)}(\mathbf{q}_0) . \quad (10)$$

The position of the unit vectors $\mathbf{q}_0$, $\mathbf{e}_1(\mathbf{q}_0)$, and $\mathbf{e}_2(\mathbf{q}_0)$ is shown in Fig. 2.

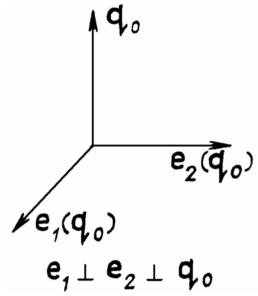

FIG. 2.

The proofs of Eqs. (9) and (10) were apparently first represented in Ref. 9; the derivation in ample detail was repeated in Ref. 10.

The important consequence of Theorems 2 and 3 (see Eqs. (8)–(10)) is the fact that at any distance from the particle to its boundary $E^{(s)}$ is determined by the same amplitudes as in the far field. This is, of course, not more than the explicit form of the remarkable result of the theory of scattering: the asymptotic solution in the far field for the particle can be used to reconstruct the general solution (for example, the abstract operator interpretation can be found in Ref. 11). On the basis of Eq. (9) it is simply interpreted: Eq. (8) follows from Eq. (9) in accordance with asymptote (5).

Theorem 4. The internal field is

$$\tilde{\mathbf{E}}(\mathbf{r}') = \frac{1}{4\pi}\frac{c^2}{\omega^2}\hat{\varepsilon}^{-1}\mathrm{rot}'\int \mathrm{d}\sigma\{(\mathbf{n}(\mathbf{r}'')\times\mathbf{E}^{(\sigma)}(\mathbf{r}''))\times$$

$$\times\,\mathrm{rot}''\hat{G}(\mathbf{r}''|\mathbf{r}') + (\mathbf{n}(\mathbf{r}'')\times\mathrm{rot}''\mathbf{E}^{(\sigma)}(\mathbf{r}''))\,\hat{G}(\mathbf{r}''|\mathbf{r}') +$$

$$+\,(\mathbf{n}(\hat{\varepsilon}\mathbf{E}^{(\sigma)}))\,\mathrm{div}''\hat{G}(\mathbf{r}''|\mathbf{r}')\} . \quad (11)$$

Here σ is the surface bounding the volume V with the external normal $\mathbf{n}(\mathbf{r}'' \in \sigma)$ and $\mathbf{E}^{(\sigma)}$ is the field on σ being the surface of the interface. The symbol "^" stands for "affinor"; in particular, $\hat{\varepsilon}$ is the affinor with the elements $\varepsilon_{\alpha\beta}$, $\hat{\varepsilon}^{-1}$ is the inverse affinor.

The particular solution of the equation for the affinor Green's function $\hat{G}$ is principal for the entire space. In case of Eq. (11) this equation is

$$\Delta\hat{G}(\mathbf{r}|\mathbf{r}') + \frac{\omega^2}{c^2}\hat{\varepsilon}\,\hat{G} + (\hat{\varepsilon} - \hat{I})\,\mathrm{grad\,div}\hat{G} = -4\pi\mathrm{rot}\hat{I}\delta(\mathbf{r}-\mathbf{r}') \quad (12)$$

($\hat{I}$ is the unit affinor).

From the formal point of view the proof of Eq. (11) is the application of the affinor version of the Green's theorem to the vector field satisfying Maxwell's equations (the corresponding technique is presented, for example, in Refs. 2 and 6).

The choice of $\hat{G}$ is not unique and is dictated by the purely pragmatic considerations of the possibility to find $\hat{G}$

for the physical situation under study; the form of Eq. (11) depends on these considerations too. The term with grad div in Eq. (12) contains two parameters: the factor of "optical inhomogeneity" $(\varepsilon_{\alpha\beta} - \delta_{\alpha\beta})$ and the factor of anisotropy. These quantities very often play the role of the "parameters of expansion" of physical solutions. Therefore, "inconvenient" grad div $\hat{G}$ can be omitted in Eq. (12) without worsening the accuracy of approximation. It becomes clear that the term grand div amplifies the corresponding large parameter in the case of large particles.

But one circumstance is absolutely general: the particular solutions for the entire space must appear, this eliminates the search for the general solution of homogeneous problem (12) with corresponding boundary conditions. The physical meaning of this is well known and is as follows: the Huygens–Fresnel principle[2,3,6] is realized only in such a way.

Theorem 5. In the case of the analytic continuation for ω ($\omega \to z = \omega + i\,\xi$), the amplitude of the scattered wave (in Eqs. (8) and (10)) is the regular function decreasing exponentially as $|z| \to \infty$ in the upper semiplane ($\xi > 0$). Field (9) possesses the same feature.

The proof repeats, to a considerable degree, the analysis of properties of $\varepsilon_{\alpha\beta}(z)$ made in Refs. 12 and 13, completed by formulas (6) and (9): the functions of z entering into $\tilde{E}$ are integer due to the finiteness of V.

The consequences of Theorem 5 are evident: the ordinary dispersion relations between Re and Im of the amplitudes of scattering.

Now, we can go over to the proof of the existence of Eq (1). Actually, the direct substitution of Eqs. (4) and Eq. (11) into Π in Eq. (8) will express the scattering amplitudes in terms of $\mathbf{E}^{(\sigma)}$. Further, the boundary conditions of electrodynamics can be used to express $\mathbf{E}^{(\sigma)}$ based on Eq. (2), since optics has no problem with discontinuous functions on the boundaries, therefore, $\mathbf{E}^{(\sigma)} = \mathbf{E}_0 + \mathbf{E}^{(s)}$ on σ. In what follows, the same amplitudes of scattering are introduced by the formulas (9) and (10) into the derived relation. Now we must study the appearing integral in order to identify those typical points which determine its values.

The above–considered procedure leads to the integral equation for the amplitudes or, if so desired, for $\mathbf{E}^{(s)}$ and this procedure can replace the "two–stage" procedure described by Eqs. (6) and (7). The free term of this equation appears from Eq. (3) and its contribution to $\tilde{E}$, calculated according to Eq. (11), is

$$\frac{c^2}{\omega^2}\hat{\varepsilon}^{-1}\,\mathrm{rot}'\left\{\mathrm{rot}'\mathbf{E}_0(\mathbf{r}') + \frac{1}{4\pi}\int\limits_{(V)} \mathrm{d}\mathbf{r}\,\frac{\omega^2}{c^2}\,(\mathbf{E}_0(\mathbf{r})\cdot(\hat{\varepsilon} - \hat{I})\times \hat{G}(\mathbf{r}|\mathbf{r}') - \frac{1}{4\pi}\int\limits_{(V)} \mathrm{d}\mathbf{r}\,\mathrm{div}\,(\hat{\varepsilon}\mathbf{E}_0)\times \mathrm{div}\,\hat{G}(\mathbf{r}|\mathbf{r}')\right\}, \qquad (13)$$

where the symbol "×" denotes the direct production.

4. LARGE PARTICLES ($l\,\omega/c \gg 1$, l IS THE PARTICLE SIZE)

Commenting on Eq. (12) we can add that the asymptotic behavior typical of the case under consideration allows one to differentiate $\exp(i\,\omega/c)\,|\mathbf{r} - \mathbf{r}'|$ alone when the expressions with the Green's functions are differentiated. Then, representing the affinor in the standard form $\hat{G} = \sum\limits_{\alpha} \mathbf{e}'_\alpha \mathbf{G}_\alpha$ we can derive

$$\mathbf{G}_\alpha = i\,\frac{\omega}{c}\sqrt{\varepsilon_\alpha}\,Q_\alpha\cdot(\mathbf{R}_0\times \mathbf{e}'_\alpha)\,,\quad Q_\alpha = \frac{\exp\left(i\,\frac{\omega}{c}|\mathbf{r} - \mathbf{r}'|\right)}{|\mathbf{r} - \mathbf{r}'|}\,, \qquad (14)$$

where $\mathbf{R}_0$ is the unit vector of the vector $\mathbf{r} - \mathbf{r}'$ and ε_α are the eigenvalues of the tensor $\varepsilon_{\alpha\beta}$; Eq. (14) is written in the primed system of coordinates shown in Fig. 1. All terms with div$\hat{C}$ can be ignored in this approximation either.

Let us now consider the term Π which arises after substituting $\mathbf{E}^{(s)}$ from Eq. (2) in the form given by Eq. (9) into the chain of transformations described at the end of Section 3. The existence of the large parameter ($l\,\omega/c$) opens the excellent possibility for the asymptotic estimate of the integrals.

First, the asymptotic integration over r' in Eqs. (8) and (4) (on account of Eq. (14) and for corresponding $F(R_0)$) is performed following the procedure invented in Refs. 14 and 15

$$\int\limits_{(V)} \mathrm{d}\mathbf{r}'\,\exp\left[-\,i\,\frac{\omega}{c}(\mathbf{r}_0 - \mathbf{r}')\right] F(\mathbf{R}_0)\,\frac{\exp\left[i\,\frac{\omega}{c}\sqrt{\varepsilon_\alpha}\,|\mathbf{r}' - \mathbf{r}''|\right]}{|\mathbf{r}' - \mathbf{r}''|} \simeq$$

$$\simeq \frac{2\pi i}{(\omega/c)}\,F(\mathbf{r}_0)\,\exp\left[-\,i\,\frac{\omega}{c}\,(\mathbf{r}_0\cdot\mathbf{r}'')\right] g_\alpha(\mathbf{r}'', \mathbf{r}_0)\,; \qquad (15)$$

$$g_\alpha(\mathbf{r}'', \rho_0) = \int\limits_0^\infty \mathrm{d}\rho\,\nu(\mathrm{r}'' + \rho\rho_0)\,\exp\left[i\,\frac{\omega}{c}\rho(\sqrt{\varepsilon_\alpha} - 1)\right] = -\,\frac{1}{2i}\,\frac{c}{\omega}\,\frac{1}{(\sqrt{\varepsilon_\alpha} - 1)}\,.$$

In Eq. (15) $\nu(\mathbf{r})$ is equal to 1 and 0, for $\mathbf{r} \in V$ and $\mathbf{r} \in \bar{V}$. The value of g_α is described by the second relation when $\mathbf{r}'' \in \sigma$ and the unit vector ρ_0 is directed "outside of the particle"; only such variants are considered in the subsequent analysis.

For the subsequent simplifications formulas (5), (8), (9), (11), (14), and (15) yield the expression written in the primed system of coordinates

$$\frac{ic}{16\pi\omega}\sum_{\alpha=1}^{3}\mathrm{e}'_\alpha\,\frac{\varepsilon_\alpha - 1}{\varepsilon_\alpha}\int \mathrm{d}\sigma \int\limits_{(\mathbf{q}_0\mathbf{r}''_0)\,\geq 0} \mathrm{d}\mathbf{q}_0\,\exp\left[-\,i\,\frac{\omega}{c}\,\mathbf{r}_0\mathbf{r}'' + i\,\frac{\omega}{c}\,\mathbf{q}_0\mathbf{r}''\right]\times$$

$$\times\,\{(\mathbf{r}_0\times(\mathbf{r}_0((\mathbf{n}\times(\mathbf{q}_0\times(\mathbf{q}_0\times\mathbf{B})))\times\mathbf{r}_0)\,\hat{J}))_\alpha -$$

$$-\,(\mathbf{r}_0\times(\mathbf{r}_0((\mathbf{n}\times(\mathbf{q}_0\times(\mathbf{q}_0\times(\mathbf{q}_0\times\mathbf{B}))))\,\hat{K}))_\alpha\}\,. \qquad (16)$$

The diagonal affinors in the primed system of coordinates containing the terms $\varepsilon_\alpha\sqrt{\varepsilon_\alpha}/(\sqrt{\varepsilon_\alpha} - 1)$ and

$\varepsilon_\alpha/(\sqrt{\varepsilon_\alpha}-1)$ are denoted by $\hat{J}$ and $\hat{K}$ and $(\ldots)_\alpha$ means α component of the vector enclosed in the parentheses.

The large parameter enters into Eq. (16) because $\mathbf{r}'' \in \sigma$. However, the strategy of the asymptotic estimate requires a preliminary explanation.

The point is this the term $\exp(-i(\omega/c)\mathbf{r}_0\mathbf{r}'')$ cannot be ignored when taking $\int d\mathbf{q}_0$ because $\int ds$ will be performed next. But the four–dimensional integral is hardly calculated immediately: first, the integration limits are dependent; second, $r'' = r''(\theta'', \varphi'')$ and the effect of this function should be avoided as long as possible; and, third, the system of equations for stationary points will be of the fourth order.

We find the way out in going over from $\int d\mathbf{q}_0$ to $\int d\mathbf{q}$ (where $\mathbf{q}_0$ is the unit vector of $\mathbf{q}$) that will allow us to introduce the variable $q\,(\omega/c)\mathbf{r}_0 = \mathbf{p}_0$. It is technically obtained by taking $\delta(q^2-\omega^2/c^2)$ and the Heaviside function with the argument $\mathbf{q}_0\mathbf{r}_0'' \to \mathbf{q}\mathbf{r}''$ inside $\int dq$. For the new variable $\mathbf{p}$ the above procedures are performed in the reverse order and $\int d\mathbf{q}_0$ in Eq. (16) is replaced by

$$4\int d\mathbf{p}_0|\mathbf{p}_0\mathbf{r}_0|\,\exp\left[-2i\frac{\omega}{c}(\mathbf{p}_0\mathbf{r}_0)(\mathbf{p}_0\mathbf{r}_0'')\right]H(\mathbf{c}_0)\,, \qquad (17)$$

$$\mathbf{c}_0 = \mathbf{r}_0 - 2\mathbf{p}_0(\mathbf{p}_0\mathbf{r}_0)\,,$$

where $\mathbf{p}_0$ is the unit vector of the vector $\mathbf{p}$, $\mathbf{c}_0$ is the unit vector, and $(\mathbf{p}_0\mathbf{r}_0) < 0$, $-2(\mathbf{p}_0\mathbf{r}_0)(\mathbf{p}_0\mathbf{r}_0'') + \mathbf{r}_0\mathbf{r}_0'' \ge 0$.

The function H is formed by $\{\ldots\}$ from Eq. (16) and by the factors being before $\int ds$.

Now we must estimate the contribution of stationary points into expression (17) and to integrate this expression over σ (the cycle "A"); the same procedure (the cycle "B") should be performed for the "boundary" points having preliminary found out their meaning (for this reason they are used in the quotation marks).

Calculations in the cycle "A" are quite standard. As it becomes clear, the only stationary point satisfying expression (17) is $\mathbf{p}_0 = -(\mathbf{r}_0 - \mathbf{r}_0'')/|\mathbf{r}_0 - \mathbf{r}_0''|$ and its contribution to expression (17) is

$$-\frac{2\pi i}{r''(\omega/c)}H(r_0'')\exp\left[i\frac{\omega}{c}(1-\mathbf{r}_0\mathbf{r}_0'')\right]. \qquad (18)$$

Asymptotic integrating of expression (18) over σ can be quite simple if the dependence of r'' on θ'' and φ'' is ignored in searching for the stationary point, then $\mathbf{r}_0$ is appeared to be such a point. Here two obviously encouraging points should be noted: omissions of exp from Eq. (18) and r'' from $d\sigma$. The point is simply that r'' depends on the choice of the start of the count (Fig. 1) but it is physically regulated by nothing. Therefore such an uncertain quantity as r'' must be omitted in the final formulas. But, nevertheless, the same uncertainty remained for $\mathbf{n}$ at the point at which the direction of $\mathbf{r}_0$ crosses σ. We eliminate this uncertainty (on the basis of the asymptote) by means of the "virtual" conversion to the volume integral ($\int(\mathbf{n}\times\mathbf{b})d\sigma = \int d\mathbf{r}\,\mathrm{rot}\mathbf{b}$) and, finally, taking the limit $\mathbf{n} \to \mathbf{r}_0$. The estimate of $\int ds$ is standard, and for functions (18) and (16) it is

$$\int d\sigma f(\mathbf{r}_0'')\exp\left[i\frac{\omega}{c}r''(1-\mathbf{r}_0\mathbf{r}_0'')\right] \simeq \frac{2\pi i r''}{(\omega/c)}f(\mathbf{r}_0)\,. \qquad (19)$$

The termination of the cycle "A" is the reconstruction of factors according to expressions (16), (18), and (19), the conversion from Π to the amplitudes according to formulas (8) and (10), and the calculations of the vectors from expression (16) ($\mathbf{q}_0 = \mathbf{n} = \mathbf{r}_0$). In what follows, the spherical system of coordinates is used (see Fig. 1) with the designations

$$\mathrm{e}_\theta = g_1,\ \mathrm{e}_\varphi = g_2,\ \mathrm{e}_r = \mathbf{r}_0 = g_3\,;$$

$$A_\theta = A_1,\ A_\varphi = A_2,\ A_r = A_3\,. \qquad (20)$$

Contribution of stationary point (17) to the amplitude of scattering is determined by the vector

$$-\frac{1}{8}\hat{N}((g_1A_1(\mathbf{r}_0) + g_2A_2(\mathbf{r}_0))(\hat{J}+\hat{K})\,. \qquad (21)$$

The definition of the affinors $\hat{J}$ and $\hat{K}$ are given after expression (16), and the affinor $\hat{N}$ being diagonal in the primed system of coordinates has the elements $(\varepsilon_\alpha - 1)/\varepsilon_\alpha$.

Rather ample foreword must precede the cycle "B" consisting in the estimate of the effect of "boundary" points.

Though, the reason is quite evident: for large particles "forward scattering" (i.e., in the direction of the unit vector $\mathbf{k}_0$, see Eq. (3) and Fig. 1) has rather sharp maximum (the examples can be found in Ref. 16), and this maximum should be represented in asymptotic estimate (17).

However, the oscillating functions cannot be ignored in any way and we will even assume that they are predominating as previously. This assumption is very important since it enables us to avoid the calculations of the derivatives of the scattering amplitudes with respect to the angles of the unit vector $\mathbf{r}_0$.

As it becomes clear, the result of this estimate is comparable with relation (21) and therefore the impression can be formed that any arbitrary point can be introduced analogously into the final formula. But we have to recall our assumption and the trick proposed below can be used to formalize the situation.

The point of the "amplitude maximum" is claimed as "boundary", i.e., the point from which the count of the angle is started; by the way, this entirely corresponds to the canons of the asymptotic analysis[17,18] where the contributions to the integral estimate comes from the stationary and boundary points. In our case the "physical maximum" amplifies this effect (the "upper" or "lower" boundary is unimportant for the two–dimensional integrals). Some mathematical details and estimates are presented in Appendix I.

Once more purely tactical argument (though, it is absolutely natural) appears during calculations: the third axis is exceptional and that sounds as a signal that such a direction should be also exceptional physically. Strictly speaking, this is reached by the choice of the system of coordinates without primes in Fig. 1 (let us recall that it is affixed to field (3) incident from outside, and now the "forward" scattering is considered).

The boundary point is formally determined by the equation $\mathbf{c}_0 = \mathbf{k}_0$, where $\mathbf{c}_0$ is given by Eq. (17). The solution of Eq. (17) is $\mathbf{p}_0 = (\mathbf{k}_0 - \mathbf{r}_0)/\sqrt{2(1 - \mathbf{k}_0\mathbf{r}_0)}$; $\mathbf{k}_0\mathbf{r}''_0 \geq 0$ and $\mathbf{r}_0 \neq \mathbf{k}_0$ in the subsequent $\int \mathrm{d}s$ since the limit of the above–written value of $\mathbf{p}_0$ as $\mathbf{r}_0 \to \mathbf{k}_0$ does not exist.

The cycle "B" is technically absolutely clear. The contribution of the boundary point to expression (17) considered as a function of $\mathbf{r}''_0$ is concentrated near $\theta'' = 0$ (the system of coordinates without primes in Fig. 1); beyond this region the function is asymptotically small. (That, by the way, makes two procedures, i.e., estimating only the stationary points in expression (16) and introducing the "boundary" for $\int \mathrm{d}s$ and another procedure being considered now identical to each other.) When $\theta'' = 0$, the value of this function is

$$-\frac{i\,\pi}{2r''(\omega/c)}\exp\left[i\,\frac{\omega}{c}\,r''(1 - \cos\theta)\right]. \tag{22}$$

The subsequent integration $\int \mathrm{d}s$ is also standard, and we eliminate $\mathbf{n}$ following the previously described procedure. The expression appear

$$\frac{2\pi i\,(\mathbf{k}_0 - \mathbf{r}_0)}{(1 - \cos\theta)^2(\omega/c)} \tag{23}$$

which must be multiplied by expression (22) after eliminating $1/\mathbf{r}''$.

The termination of the cycle "B" is the list of factors from expressions (16), (22), and (23) and Eqs. (8) and (10) as well as the calculation of vectors {...} from expression (16) (after substitutions $\mathbf{q}_0 \to \mathbf{k}_0$ and $\mathbf{n} \to \mathbf{k}_0 - \mathbf{r}_0$) where the system of notation (20) is used. The term

$$-\frac{1}{32}\frac{1}{(1 - \cos\theta)^2}\hat{N}\,(L_1\mathbf{g}_1 + L_2\mathbf{g}_2)\,,$$

where

$$\mathbf{L} = (1 - \cos\theta)\,\mathbf{D}(\hat{K} - \hat{J}) + (\mathbf{r}_0\mathbf{D})(\mathbf{k}_0(\hat{K} - \hat{J}) + (\mathbf{r}_0\mathbf{D})(\mathbf{r}_0\hat{J}); \tag{24}$$

$$\mathbf{D} = \mathrm{e}_1\tilde{A}_1(k_0) + \mathrm{e}_2\tilde{A}_2(k_0)$$

will be added to expression (21). Here $\tilde{A}_1$ and $\tilde{A}_2$ are the projections of $\mathbf{A}$ on the unit vectors $\mathbf{e}_1$ and $\mathbf{e}_2$ of the coordinate system without primes.

Now we must calculate the contribution of expression (13) to Π according to the chain described at the end of Section 3. It is possible to estimate the corresponding integrals asymptotically, but the result is not suitable for the inverse problems. Therefore, there is a sense in the next trick.

Equation (12) (without grad div) allows one to be convinced that in {...} from Eq. (13) rot from the first of two terms of the Neumann series of the equation is

$$\Delta\mathbf{E} + \frac{\omega^2}{c^2}\,\hat{\varepsilon}\mathbf{E} = -(\hat{\varepsilon} - \hat{I})\,\mathbf{E}\,\frac{\omega^2}{c^2}\,. \tag{25}$$

The trick just consists in the substitution of the solution of Eq. (25) instead of expression (13). Asymptotically it is constructed according to Ref. 14 (the method proposed in Ref. 14 was repeated in Ref. 19 much later) and the final contribution to the amplitude is ($\theta \neq 0$ as previously)

$$-\frac{\omega^2(1 - \cos\theta)}{2\pi c^2}\int\limits_{(S)}\mathrm{d}x'\mathrm{d}y'\,\exp\left[-\,i\,\frac{\omega}{c}(x'r_{01} + y'r_{02})\right]\times$$

$$\times\int\limits_{-\infty}^{+\infty}\mathrm{d}z'\mathbf{E}^{(0)}\hat{M}\,\exp\left[i\,\frac{\omega}{c}\,z'(1 - \cos\theta)\right]. \tag{26}$$

Here r_{01} and r_{02} are the components of $\mathbf{r}_0$ in the coordinate system without primes, $\mathbf{E}^{(0)}$ is determined by Eq. (3), S is the projection of the particle onto the plane perpendicular to $\mathbf{k}_0$ (x', y', z' are the coordinates of $\mathbf{r}' \in V$ in the coordinate system without primes), and the affinor $\hat{M}$ being diagonal in the primed system has the components $M_\alpha = \exp(i\omega/2c)(\varepsilon_\alpha - 1)\int_{-\infty}^{z'}\nu(z'')\mathrm{d}z''$ with $\nu(z') = 1$ inside V and $\nu = 0$ outside of V.

5. DISCUSSION

In the previous section we demonstrated the conversion of the general theorems presented in Section 3 into relation (1) suitable for the specific analysis: now the amplitude $\mathbf{A}(\mathbf{r}_0)$ must be equated to the sum of expressions (21), (24), and (26). The physical arguments should precede this action and in Section 4 the maximum of "forward scattering" is of primary importance.

The pragmatical sense of the derived relation can be interpreted in a quite evident way here. The retrospective glance on its derivation convinces that we must not consider the calculations of the scattering parameters in terms of other parameters in the specific way. It is assumed that measurements are available (for example, of the scattering matrix) at different angles of scattering. Using them together with the relations under discussion we can obtain the information about the value described by expression (26), and it can be then "inverted" into the particle properties.

The most interesting possibility is to express the radar cross section ($\theta = \pi$ in the discussed formulas) in terms of the amplitude of the wave scattered forward. The optical theorem and dispersive relations following from Theorem 5 can be used to express the radar cross section at the frequency ω in terms of the integral of the coefficient of extinction over ω.

Further, the estimates can (and should) be significantly improved if it is known *a priori*, for example, that a particle is isotropic or has the "regular" shape (a sphere, a definitely oriented ellipsoid or a cylinder, etc). Though any *a priori* information can be used, as it is in the inverse problems, to refine many details of the analysis. It becomes possible to perform $\int \mathrm{d}s$ exactly and to calculate expressions (26) or (13). Apparently, for strongly anisotropic particles there is a sense to refine Eq. (14) with the help of successive approximations.

The assumption preceding the estimates of expressions (22)–(24) (and it is absolutely true) is too arbitrary. There is no specific formal problems with another estimate presented in Appendix II; it can be useful especially if the particles are not very large. However, certain additional difficulties can appear in the inverse problem.

Note also that the solution of Eq. (25) (see Appendix II) will be the zeroth order approximation of the equation for $\mathbf{E}^{(s)}$ which was discussed at the end of Section 3. It is well–known in the theory of scattering[20] that these results excelently imitate the precise results and for this reason, our trick, when we go over to Eq. (25), receives the additional support. We hope that already next iteration will become comprehensive (Some details are given in Appendix III).

APPENDIX I

To explain the trick yielding expression (24), let us consider the integral

$$J = \int_{a_1}^{a_2} dx \int_{b_1}^{b_2} dy g(x, y) f(x, y) \exp[i\lambda h(x, y)] . \qquad (27)$$

The function $g(x, y)$ varies quite slowly and the function $f(x, y)$ has maximum at the "lower boundary" (a_1, b_1) and represents the modulus of the amplitude of scattering (the argument of the complex amplitude can be assumed to be included in g). The parameter $|\lambda| \gg 1$ and the real function h has no extremum at the lower boundary. The following quantities are nonzero:

$$\left(\frac{\partial h}{\partial x}\right)_{a_1b_1} = \sigma_1 , \left(\frac{\partial h}{\partial y}\right)_{a_1b_1} = \sigma_2 . \qquad (28)$$

The calculation of the contribution of the boundary (a_1, b_1) to Eq. (27) becomes standard after formal transformation $f(x, y) = \exp(\ln f(x, y)) = \exp H(x, y)$ and then

$$b_{11} = \left(\frac{\partial^2 H}{\partial x^2}\right)_{a_1b_1} = \left(\frac{1}{f}\frac{\partial^2 f}{\partial x^2}\right)_{a_1b_1} ;$$

$$b_{12} = \left(\frac{\partial^2 H}{\partial x \partial y}\right)_{a_1b_1} = \left(\frac{1}{f}\frac{\partial^2 f}{\partial x \partial y}\right)_{a_1b_1} ; \qquad (29)$$

$$b_{22} = \left(\frac{\partial^2 H}{\partial y^2}\right)_{a_1b_1} = \left(\frac{1}{f}\frac{\partial^2 f}{\partial y^2}\right)_{a_1b_1} ,$$

since, according to our assumption, the derivatives $\partial f / \partial x = \partial f / \partial y = 0$ in the interval $(a_1\ b_1)$.

After conventional in such cases expanding H and h in the Tailor series up to the first nonzero derivatives, substituting the variables $\xi = x - a_1$ and $\eta = y - b_1$, and integrating between the semiinfinite limits, since the considered combination (a_1, b_1) is boundary, the expression appear

$$g(a_1, b_1) f(a_1, b_1) \exp[i \lambda h(a_1, b_1)] \int_0^\infty d\xi \int_0^\infty d\eta \ \times$$

$$\times \exp\left(i \lambda\sigma_1\xi + i \lambda\sigma_2\eta - \frac{1}{2}\left(|b_{11}|\xi^2 - 2b_{12}\xi\eta + |b_{22}|\eta^2\right)\right), \qquad (30)$$

where the negatively defined quadratic form with coefficients (29) has been taken into account.

The integral, for example, over ξ is

$$\int_0^\infty \exp(-\beta\xi^2 - \gamma\xi)(\cos b\xi + i \sin b\xi) d\xi = K_c + iK_s$$

with $\beta > 0$ (as has already been noted) and $b > 0$ (otherwise the sign of K_s is reversed) is tabular:

$$K_c = 1/4 \sqrt{\pi/\beta} \left\{\left(1 - \Phi((\gamma - ib)/2\sqrt{\beta})\right)\exp\left[(\gamma - ib)^2/4\beta\right] + \right.$$

$$\left. + \left[1 - \Phi\left((\gamma + ib)/2\sqrt{\beta}\right)\right]\exp\left[(\gamma + ib)^2/4\beta\right]\right\} .$$

The expression for K_s differs from the expression for K_c in the "minus" sign between the terms in {...} and in the factor i being before {...}; according to the definition,

$$\Phi(z) = 2/\sqrt{\pi} \int_0^z \exp(-t^2) dt .$$

If we assume that

$$|b| \gg \sqrt{\beta} \text{ and } |b| \gg |\gamma| , \qquad (31)$$

then, applying the "large $|z|$" asymptote to $\Phi(\zeta)$, after elementary transformations we obtain that $K_s = 1/b$. The quantity $K_c \simeq 0$ given that condition (31) is valid, and this circumstance is quite noteworthy. If the point of maximum f were internal, the lower limit in expression (30) would be $(-\infty)$, and the evident conversion to the sum of squares in the form with coefficients (29) would have yielded $K_s = 0$ and double K_c. Given that conditions (31) be valid, the contribution of the maximum f would be equal to zero (!?).

The rest of integral (30) is

$$\int_0^\infty \exp(-a\eta^2) dt\ (\cos b'\eta + i \sin b'\eta) d\eta = N_c + iN_s$$

with $a > 0$ and $b' > 0$ (as before). The written integrals are tabular

$$N_c = 1/2 \sqrt{\frac{\pi}{a}} \exp(-b'^2/4a);$$

$$N_s = \frac{b'}{2a} \Phi\left(\frac{1}{2}, \frac{3}{2}; \frac{b'^2}{4a}\right) \exp(-b'^2/4a) .$$

Let us assume again that

$$b'^2 \gg a \qquad (32)$$

and once more $N_c \simeq 0$ (with the previous comment). The asymptote of the degenerate hypergeometric function Φ will give $N_s \simeq 1/b'$. This is entirely equivalent to the estimate of K_s.

Reconstructing the values of b and b' we can obtain the contribution of (a_1, b_1) to Eq. (27)

$$-\frac{1}{\lambda^2} \frac{1}{\left(\frac{\partial h}{\partial x}\right)_{a_1b_1} \left(\frac{\partial h}{\partial y}\right)_{a_1b_1}} f(a_1, b_1)\, g(a_1, b_1) \exp[i \lambda h(a_1, b_1)].$$

But absolutely the same result can be obtained if Eq. (27) is estimated by the conventional asymptotic technique[17] employed for f without extremum, that is, by the integration by parts ignoring the upper limit when making substitutions in the free term. It is easy to prolong this rule also for more complicated variants with zero derivatives $\partial h/\partial x$ or $\partial h/\partial y$ and even with the extremum in h on the boundary.

In terms of (28) and (29), conditions (31) and (32) claim that at the "boundary" point

$$\left|\lambda\frac{\partial h}{\partial x}\right| \gg \sqrt{\left|\frac{1}{f}\frac{\partial^2 f}{\partial x^2}\right|} \text{ and } \sqrt{\left|\frac{1}{f}\frac{\partial^2 f}{\partial x\,\partial y}\right|}\ ,$$

$$\left|\lambda\frac{\partial h}{\partial y}\right| \gg \sqrt{\left|\frac{1}{f}\frac{\partial^2 f}{\partial y^2}\right|} \text{ and } \sqrt{\left|\frac{1}{f}\frac{\partial^2 f}{\partial x\,\partial y}\right|}\ , \quad (33)$$

and they express mathematically the assumption which was made before accounting for the physical maximum of the "forward scattering". Its mathematical content becomes clear: the large values of $|\lambda|$ must quarantee the validity of conditions (33), thereby the maximum of f becomes only the "background" for the fast oscillations of $\exp(i\lambda h)$.

APPENDIX II

The procedure proposed in Ref. 14 which has already been mentioned about can be used to derive the physically clear solution of Eq. (25) for the internal field:

$$\tilde{E}(\mathbf{r}') = \sum_{\alpha=1}^{3} \mathbf{e}'_\alpha\, \mathbf{E}^{(0)}_\alpha \exp\left[i\frac{\omega}{c}\,\mathbf{k}_0\mathbf{r}' + i\frac{\omega}{2c}\int_0^\infty \nu(\mathbf{r}' - \rho\mathbf{k}_0)\mathrm{d}\rho\right]. \quad (34)$$

The primed system of coordinates and the designations from Eq. (3) and expression (26) have been used here.

After substituting Eq. (34) into Eq. (8) and introducing $\nu(\mathbf{r}')$ into the integral for Π, the vector Π for $\mathbf{r}_0 = \mathbf{k}_0$ will have the components proportional to

$$\frac{ic}{2\pi\omega}\int_{(S)} \mathrm{d}x'\mathrm{d}y'\left(1 - \exp\left[i\frac{\omega}{2c}(\varepsilon_\alpha - 1)\times\right.\right.$$

$$\left.\left.\times\left(z^{(+)}(x', y') - z^{(-)}(x', y')\right)\right]\right) \quad (35)$$

with the designations used in expression (26). (In fact, this is Π for $\mathbf{r}_0 \neq \mathbf{k}_0$). The functions $z^{(+)}(x', y')$ are the solutions of the equation, describing the surface σ, for z' (the coordinates along the axis parallel to $\mathbf{k}_0$).

Now, we can demonstrate the estimate of derivatives appearing in condition (33) using the variable θ as an example (see Fig. 1). Here, we refer to that comment to Eq. (34), which was given at the end of Section 5.

Actually, it is necessary to differentiate Π along with Eq. (34) twice over θ, then to set $\theta = 0$ ("forward" scattering), and to calculate the appearing integrals following the previously described procedure (which leads to expression (26) and Eq. (34)). Then it becomes clear that

$$\frac{\partial^2 f}{\partial x^2} \to \frac{1}{2\pi}\int_{(S)}\mathrm{d}x'\mathrm{d}y'\left(z^{(+)}\exp\left[2i\frac{\omega}{c}(\varepsilon_\alpha - 1) -\right.\right.$$

$$\left.\left.-\left(z^{(+)} - z^{(-)}\right)\right] - z^{(-)}\right) - \frac{f(0)}{\varepsilon_\alpha - 1} + \frac{i\,\omega}{2c}\times$$

$$\times\int_{(S)}\mathrm{d}x'\mathrm{d}y'\,(x'^2 + y'^2)\left(1 - \exp\left[i\frac{\omega}{2c}(\varepsilon_\alpha - 1)(z^{(+)} - z^{(-)})\right]\right). \quad (36)$$

Expression (35) is denoted by $f(0)$. The ratio between expressions (36) and (35) should be taken as the estimate of Eq. (29).

APPENDIX III

Substituting subsequently Eqs. (4), (11), and (2) into Eq. (7) instead of $\mathbf{E}^{(\sigma)}$, we can derive the integral equation directly for $\mathbf{E}^{(s)}$. We do not write the term with div$''\hat{G}$ again because this approximation has been already commented (though, it is absolutely unimportant here). Then

$$\mathbf{E}^{(s)}(\mathbf{r}) = \mathbf{E}^{(s)}_0(\mathbf{r}) - \frac{1}{16\pi^2}\frac{c^2}{\omega^2}\,\text{rot rot}\int_{(V)}\mathrm{d}\mathbf{r}'\; G(\mathbf{r}\,|\,\mathbf{r}')(\hat{\varepsilon} - \hat{I})\cdot\hat{\varepsilon}^{-1}\times$$

$$\times\,\text{rot}'\int \mathrm{d}s\,\left\{(\mathbf{n}(\mathbf{r}'')\times\mathbf{E}^{(s)}(\mathbf{r}''))\text{rot}''\hat{G}(\mathbf{r}''\,|\,\mathbf{r}') + \right.$$

$$\left. + (\mathbf{n}(\mathbf{r}'')\times\text{rot}''\mathbf{E}^{(s)}(\mathbf{r}''))\hat{G}(\mathbf{r}''\,|\,\mathbf{r}')\right\}. \quad (37)$$

The free term $\mathbf{E}^{(s)}_0$ is the combination of Eq. (7) and (4) into which Eq. (34) is substituted for $\tilde{\mathbf{E}}$. This procedure, described at the end of Section 5, quarantes the effectiveness of Eq. (37) for the particles whose size varies within wide limits.

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