EFFECTIVE RADIATIVE TRANSFER PROPERTIES FOR PARTIALLY CLOUDY ATMOSPHERES

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In this paper, we suggest a relatively simple radiation treatment of the broken cloud field problem. The underlying equations are based upon a Markovian model, which treats this radiative transfer problem by a stochastic formalism. Specifically, the partially cloudy atmosphere is treated as a two-component (clouds and clear sky) mixture, which is described by a set of two coupled deterministic equations for the ensemble-averaged intensity. For Markovian statistics, these equations are exact in a purely absorbing mixture, and are reasonably accurate and very robust in the general case including scattering. This description can also be modified to account for non-Marcovian statistics. We show that in two different asymptotic limits this set of two coupled equations can be renormalized to a single radiative transfer equation, involving effective atmospheric properties. These two limits correspond to a nearly transparent atmosphere and small correlation length mixing statistics. General spatial dependences of cloud and clear sky properties, as well as inhomogeneous and anisotropic statistics, are allowed in the formalism.

1. INTRODUCTION

It is widely accepted that the interaction of thermal radiation with clouds is an important factor in the determination of the earth's climate. Such considerations must be incorporated into general circulation models of the atmosphere if these models are to predict accurately long-term climate trends. Discussion of this can be found in the papers of Stephens,[1] Ramanathan et al.,[2] Stephens et al.,[3] and the references therein. The pertinent equation of radiative transfer in this context is

$$\mathbf{\Omega}\cdot\nabla I(\mathbf{r}, \mathbf{\Omega}) + \sigma'(\mathbf{r})I(\mathbf{r}, \mathbf{\Omega}) = LI(\mathbf{r}, \mathbf{\Omega}) + S(\mathbf{r}), \qquad (1)$$

where L is the scattering operator defined by

$$LI(\mathbf{r}, \mathbf{\Omega}) = \sigma'_s(\mathbf{r}) \int\limits_{4\pi} \mathrm{d}\mathbf{\Omega}' f(\mathbf{r}, \mathbf{\Omega}'\cdot\mathbf{\Omega}) I(\mathbf{r}, \mathbf{\Omega}), \qquad (2)$$

and $S(\mathbf{r})$ is the (isotropic) emission source given by

$$S(\mathbf{r}) = \sigma_a(\mathbf{r})B[T(\mathbf{r})]. \qquad (3)$$

Here $I(\mathbf{r}, \mathbf{\Omega})$ is the specific intensity of radiation at spatial point $\mathbf{r}$ in the direction $\mathbf{\Omega}$, $\sigma'_s(\mathbf{r})$ is the scattering cross section, $\sigma_a(\mathbf{r})$ is the absorption cross section corrected for induced emission, $\sigma'(\mathbf{r}) = \sigma'_s(\mathbf{r})+\sigma_a(\mathbf{r})$, and $B[T(\mathbf{r})]$ is the Plank function at temperature $T(\mathbf{r})$. The function $f(\mathbf{r}, \mathbf{\Omega}'\cdot\mathbf{\Omega})$ describes the redistribution in the angle associated with the scattering process, and has the normalization

$$\int\limits_{4\pi} \mathrm{d}\mathbf{\Omega}' f(\mathbf{r}, \mathbf{\Omega}'\cdot\mathbf{\Omega}) = 2\pi \int\limits_{-1}^{1} \mathrm{d}\xi f(\mathbf{r}, \xi) = 1. \qquad (4)$$

The physics of Eqs. (1) through (4) is time-independent radiative transfer in an isotropic medium assumed to be in local thermodynamic equilibrium, with photon scattering taken as conservative (a photon does not change frequency upon scattering). Equation (1) is thus a monochromatic equation of transfer, valid at each frequency ν. For simplicity of notation, we have not indicated this frequency variable that is simply a parameter in Eq. (1). Henceforth, we shall also drop, again for simplicity of notation, the spatial variable $\mathbf{r}$ in the argument list of all functions.

The problem in using Eq. (1) directly in the broken cloud field context is twofold. First, the size of any individual cloud is generally much smaller than the grid size in a typical general circulation model numerical simulation. Thus, one is faced with a subgrid modeling problem. Second, the location and geometry of each individual cloud are not known in the usual deterministic sense. Thus, Eq. (1) is a stochastic equation of transfer, with the four atmospheric radiative transfer properties appearing in this equation, namely σ', σ'_s, f, and S, all random variables. If we consider the broken cloud field as a

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stochastic mixture of two immiscible materials, namely clouds and clear sky, these properties are discrete, two-state random variables. It then follows that the specific intensity of radiation I is a (continuous) random variable, and the primary quantity of interest is the ensemble-averaged intensity $\langle I \rangle$.

Treating the cloud-radiation interaction problem by a statistical formalism has been suggested by several authors, including Titov,[4] Stephens et al.,[3] Malvagi et al.,[5] and Malvagi and Pomraning.[6] One relatively simple model, whose origin is in the linear transport and kinetic theory community,[7] has in particular been suggested to compute $\langle I \rangle$. This model, which is based upon Markovian mixing of two atmospheric components (clouds and clear sky), is in the form of two coupled differential equations of radiative transfer. An integral formulation of this same model has been proposed by Titov.[4] This model, while a relatively simple deterministic replacement for the stochastic description given by Eq. (1), is still complex from a calculational viewpoint. It involves two coupled equations that must be solved simultaneously to find the ensemble-averaged intensity $\langle I \rangle$. The purpose of this paper is to show that under certain circumstances, these two coupled equations can be reduced to a single renormalized equation of radiative transfer of the conventional form given by Eq. (1), but with effective, deterministic atmospheric properties $\sigma'_{\rm eff}$, $L_{\rm eff}$, and $S_{\rm eff}$. These effective parameters incorporate the properties of each atmospheric component (clouds and clear sky) at each space point $\mathbf{r}$, as well as the statistics of this two-component mixture (cloud size and spacing information). The properties of each component of this binary mixture are allowed arbitrary spatial dependences, and the mixing statistics are allowed to be inhomogeneous (space dependent) and anisotropic (direction dependent).

The two circumstances under which renormalization is possible correspond to two distinct asymptotic limits, namely: (1) the nearly transparent limit and (2) the small correlation length limit. In the first case, we envision a small amount of relatively opaque material (all cross sections large) admixed with a large amount of relatively transparent material (all cross sections small). In the second case, we envision a small correlation length problem, which implies clouds and/or cloud spacing, measured in photon mean free paths. Appropriate scalings are introduced into the Markovian two-equation model to reflect each of these two physical situations, and asymptotic expansions then give a single renormalized equation of transfer in each instance.

The remainder of this paper is organized as follows. The next section discusses the two-coupled-equation model for $\langle I \rangle$ which forms the starting point for our renormalization analysis. As mentioned earlier, arbitrary spatial dependences of all quantities as well as directionally dependent statistics are allowed. In this section, we also introduce the transport cross section philosophy that removes anisotropic scattering from Eq. (1). The two-equation deterministic model for this simplified isotropic scattering stochastic equation of transfer is also discussed. Section 3 develops the asymptotic formalism associated with the nearly transparent limit, and Section 4 considers the analogous treatment for the small correlation length asymptotic limit. The final section of the paper is devoted to a few concluding remarks.

2. THE STOCHASTIC MODEL

The ensemble-averaged specific intensity of radiation in a two-component stochastic mixture, with the components identified by subscripts 0 and 1 is clearly given by

$$\langle I(\mathbf{\Omega}) \rangle = p_0\, I_0(\mathbf{\Omega}) + p_1\, I_1(\mathbf{\Omega}). \tag{5}$$

Here p_i is the probability of finding material i at space point $\mathbf{r}$, and $I_i(\mathbf{\Omega})$ is the conditional ensemble average of the intensity, conditioned upon material i being at space point $\mathbf{r}$. In the case of Markovian mixing, a simple model for the I_i has been proposed by several authors, using a variety of arguments to arrive at the same model. These arguments include using the Liouville master equation,[8] employing an upwind closure to a stochastic balance equation,[9] using neutron transport noise ideas,[10] and invoking an independent path length assumption.[11] The Markovian mixing assumption is embodied in the equation

$$\mathrm{Prob}(i \to j) = \mathrm{d}s/\lambda_i(s), \ \ j \neq i. \tag{6}$$

The content of this equation is that as a photon traverses the binary mixture in the direction $\mathbf{\Omega}$ (s is a spatial coordinate in this direction), the probability Prob ($i \to j$) of making a transition from material i to material j, given that material i is present at position s, in a distance ds is simply proportional to ds, with the proportionality constant being the inverse of the Markovian transition length $\lambda_i(s)$. In general, the λ_i depend upon both $\mathbf{r}$ and $\mathbf{\Omega}$, and the $p_i(\mathbf{r})$ and $\lambda_i(\mathbf{r}, \mathbf{\Omega})$ are related by the forward form of the Chapman-Kolmogorov equations given by[7]

$$\frac{\mathrm{d}p_i}{\mathrm{d}s} = \frac{p_j}{\lambda_j} - \frac{p_i}{\lambda_i}, \ \ j \neq i. \tag{7}$$

For physical realizability, the $\mathbf{\Omega}$ dependences of the λ_i must be such that the p_i are independent of $\mathbf{\Omega}$. For homogeneous statistics, the λ_i depend only upon $\mathbf{\Omega}$. In this case $\lambda_i(\mathbf{\Omega})$ is the mean chord length of material i in direction $\mathbf{\Omega}$, and Eq. (7) yields

$$p_i = \frac{\lambda_i(\mathbf{\Omega})}{\lambda_0(\mathbf{\Omega}) + \lambda_1(\mathbf{\Omega})} \tag{8}$$

Further, in this case the chord length distribution in each material is a classic Poisson distribution in each direction $\mathbf{\Omega}$, namely exponential with mean $\lambda_i(\mathbf{\Omega})$. If the λ_i are independent of $\mathbf{\Omega}$, the statistics are said to be isotropic. However, in the broken cloud field context, it is important to retain the $\mathbf{\Omega}$ dependences of the λ_i, since clouds generally have different mean chord lengths in different directions. Our considerations allow an arbitrary $\mathbf{\Omega}$ dependence of λ_i.

The two coupled equations for the I_i, which have been suggested as a reasonable model describing radiative transfer in a binary Markovian mixture, are[7–11]

$$(\mathbf{\Omega}\cdot\nabla + \sigma_i')p_i I_i(\mathbf{\Omega}) = = L_i\, p_i\, I_i(\mathbf{\Omega}) + p_i\, S_i + \frac{p_j\, I_j(\mathbf{\Omega})}{\lambda_j(\mathbf{\Omega})} - \frac{p_i\, I_i(\mathbf{\Omega})}{\lambda_i(\mathbf{\Omega})}, \quad j \neq i. \tag{9}$$

Here the subscript i on σ', L, and S indicates that these quantities are applicable to the ith material, with $i = 0, 1$ in our binary mixture. These equations are exact for a purely absorbing mixture,[12] and quite accurate and very robust when scattering is present.[9,13] An integral form of these equations has been presented by Titov.[4] These two equations can be written in an equivalent form by making a dependent variable change from I_0 and I_1 to $\langle I\rangle$ and χ according to

$$\langle I\rangle = p_0\, I_0 + p_1\, I_1, \tag{10}$$

$$\chi = \sqrt{p_0\, p_1}\,(I_0 - I_1). \tag{11}$$

We find the two equations, making use of Eq. (7),

$$[\mathbf{\Omega}\cdot\nabla + \langle\sigma'\rangle]\langle I(\mathbf{\Omega})\rangle + \nu'\chi(\mathbf{\Omega}) = \langle S\rangle + \langle L\rangle\langle I(\mathbf{\Omega})\rangle + K\chi(\mathbf{\Omega}), \tag{12}$$

$$[\mathbf{\Omega}\cdot\nabla + \hat{\sigma}'(\mathbf{\Omega})]\chi(\mathbf{\Omega}) + \nu'\langle I(\mathbf{\Omega})\rangle = T + J\chi(\mathbf{\Omega}) + K\langle I(\mathbf{\Omega})\rangle. \tag{13}$$

Here we have defined the parameters

$$\langle\sigma'\rangle = p_0\, \sigma_0' + p_1\, \sigma_1', \tag{14}$$

$$\hat{\sigma}'(\mathbf{\Omega}) = p_0\, \sigma_1' + p_1\, \sigma_0' + \frac{1}{\lambda_c(\mathbf{\Omega})}, \tag{15}$$

$$\nu' = \sqrt{p_0\, p_1}(\sigma_0' - \sigma_1'), \tag{16}$$

$$\langle S\rangle = p_0\, S_0 + p_1\, S_1, \tag{17}$$

$$T = \sqrt{p_0\, p_1}(S_0 - S_1), \tag{18}$$

and the operators

$$\langle L\rangle = p_0\, L_0 + p_1\, L_1, \tag{19}$$

$$J = p_0\, L_1 + p_1\, L_0, \tag{20}$$

$$K = \sqrt{p_0\, p_1}\,(L_0 - L_1). \tag{21}$$

The quantity $\lambda_c(\mathbf{\Omega})$ in Eq. (15) is the correlation length for these Markovian statistics, given by[7]

$$\frac{2}{\lambda_c(\mathbf{\Omega})} = \frac{1}{p_1\, \lambda_0(\mathbf{\Omega})} + \frac{1}{p_0\, \lambda_1(\mathbf{\Omega})}. \tag{22}$$

A simplified equation of transfer often employed in radiative transfer calculations is one involving isotropic scattering. That is, Eq. (1), which describes general anisotropic scattering, is replaced with an equivalent (in a sense to be made precise shortly) isotropic scattering equation. This equation is[14,15]

$$\mathbf{\Omega}\cdot\nabla I(\mathbf{\Omega}) + \sigma I(\mathbf{\Omega}) = \frac{\sigma_s}{4\pi}\int_{4\pi} d\mathbf{\Omega}'\, I(\mathbf{\Omega}') + S, \tag{23}$$

where

$$\sigma_s = \sigma_s'\,(1 - \bar{\mu}), \quad \sigma = \sigma_s + \sigma_a. \tag{24}$$

Here $\bar{\mu}$ is the average value of the scattering cosine (the asymmetry factor) given by

$$\bar{\mu} = 2\pi\int_{-1}^{1} d\mu\, \mu\, f(\mu). \tag{25}$$

Equations (1) and (23) are equivalent in the sense that they yield the same classic diffusion equation for E, defined as

$$E = \int_{4\pi} d\mathbf{\Omega}\square I(\mathbf{\Omega}). \tag{26}$$

This quantity E is simply the product of the speed of light and the radiation energy density (per unit frequency). This common diffusion equation for E is given by[14,15]

$$-\nabla\cdot(1/3\sigma)\nabla E + \sigma_a\, E = 4\pi S. \tag{27}$$

If Eq. (23) is interpreted as a stochastic equation describing radiative transfer in a binary Markovian mixture, one has the corresponding model equations for the $I_i(\mathbf{\Omega})$ given by

$$(\mathbf{\Omega}\cdot\nabla + \sigma_i)p_i\, I_i(\mathbf{\Omega}) = \frac{\sigma_{si}}{4\pi}\int_{4\pi} d\mathbf{\Omega}' p_i\, I_i(\mathbf{\Omega}') + + p_i\, S_i + \frac{p_j\, I_j(\mathbf{\Omega})}{\lambda_j(\mathbf{\Omega})} - \frac{p_i\, I_i(\mathbf{\Omega})}{\lambda_i(\mathbf{\Omega})}, \quad j\neq i. \tag{28}$$

Under the charge of variables given by Eqs. (10) and (11), an equivalent set of equations is

$$[\mathbf{\Omega}\cdot\nabla + \langle\sigma\rangle]\,\langle I(\mathbf{\Omega})\rangle + \nu\,\chi(\mathbf{\Omega}) = = \langle S\rangle + \frac{1}{4\pi}\int_{4\pi} d\mathbf{\Omega}'\,[\langle\sigma_s\rangle\langle I(\mathbf{\Omega}')\rangle + \nu_s\,\chi(\mathbf{\Omega}')], \tag{29}$$

$$[\mathbf{\Omega}\cdot\nabla + \hat{\sigma}(\mathbf{\Omega})]\,\chi(\mathbf{\Omega}) + \nu\,\langle I(\mathbf{\Omega})\rangle = = T + \frac{1}{4\pi}\int_{4\pi} d\mathbf{\Omega}'\,[\hat{\sigma}_s\,\chi(\mathbf{\Omega}') + \nu_s\,\langle I(\mathbf{\Omega}')\rangle]. \tag{30}$$

Here $\langle S\rangle$ and T are given by Eqs. (17) and (18), and additionally we have defined

$$\langle\sigma\rangle = p_0\, \sigma_0 + p_1\, \sigma_1, \tag{31}$$

$$\hat{\sigma}(\mathbf{\Omega}) = p_0\, \sigma_1 + p_1\, \sigma_0 + [1/\lambda_c(\mathbf{\Omega})], \tag{32}$$

$$\nu = \sqrt{p_0\, p_1}\,(\sigma_0 - \sigma_1), \tag{33}$$

$$\langle\sigma_s\rangle = p_0\,\sigma_{s0} + p_1\,\sigma_{s1}, \tag{34}$$

$$\hat{\sigma}_s = p_0\,\sigma_{s1} + p_1\,\sigma_{s0}, \tag{35}$$

$$\nu_s = \sqrt{p_0\,p_1}\,(\sigma_{s0} - \sigma_{s1}). \tag{36}$$

The next two sections of this paper develop the two asymptotic limits referred to earlier, utilizing both Eqs. (12) and (13), and Eqs. (29) and (30), as the starting point. These two limits are the nearly transparent limit discussed in the next section, and the small correlation length limit treated in the section after that. In the case of Eqs. (12) and (13), we find a renormalized equation of transfer for $\langle I\rangle$ in both limits given by

$$\mathbf{\Omega}\cdot\nabla\langle I(\mathbf{\Omega})\rangle + \sigma'_{\text{eff}}(\mathbf{\Omega})\langle I(\mathbf{\Omega})\rangle =$$
$$= L_{\text{eff}}\langle I(\mathbf{\Omega})\rangle + S_{\text{eff}}(\mathbf{\Omega}), \tag{37}$$

with different effective properties for each of the two asymptotic limits. Similarly, using Eqs. (29) and (30) as the basis for the analysis, the normalized equation of transfer is written

$$\mathbf{\Omega}\cdot\nabla\langle I(\mathbf{\Omega})\rangle + \sigma_{\text{eff}}(\mathbf{\Omega})\,\langle I(\mathbf{\Omega})\rangle =$$
$$= \frac{1}{4\pi}\int_{4\pi} \mathrm{d}\mathbf{\Omega}'[\langle\sigma_{s,\,\text{eff}}(\mathbf{\Omega}',\mathbf{\Omega})\langle I(\mathbf{\Omega}')\rangle + S_{\text{eff}}(\mathbf{\Omega}). \tag{38}$$

In each asymptotic limit, we obtain explicit and relatively simple expressions for σ'_{eff}, L_{eff}, S_{eff}, σ_{eff}, and $\sigma_{s,\,\text{eff}}$. Thus, our analysis yields effective properties, accounting for the statistic nature of the problem, which are to be used in the classic deterministic equation of transfer. One complication, as can be seen from the appearance of the argument $\mathbf{\Omega}$ in the various terms in Eqs. (37) and (38), is that these effective properties involve angular dependences not present in the corresponding properties of each component of the mixture. These angular dependences arise from the angular dependences of the Markov transition lengths $\lambda_i(\mathbf{\Omega})$. In the case of isotropic statistics (λ_i independent of $\mathbf{\Omega}$), we will find that these unusual angular dependences are not present. However, in the broken cloud field context, the angular dependence of $\lambda_i(\mathbf{\Omega})$ is needed to account for the directionally dependent mean cloud chord length.

Finally, we note that the two-equation models summarized here, namely Eqs. (12) and (13), and Eqs. (29) and (30), are only strictly applicable to Markovian statistics as defined by Eq. (6). It has been suggested, however, that these models can be used in the case of certain non-Markovian statistics by modifying the correlation length $\lambda_c(\mathbf{\Omega})$ found in Eqs. (15) and (32) (see Ref. 16). This class of statistics is referred to as renewal statistics, defined by chord length distributions. If the chord length distribution for chord length τ (in direction $\mathbf{\Omega}$) in material i is denoted by $g_i(\tau)$, we define $G_i(\tau)$ as

$$G_i(\tau) = \int_{\tau}^{\infty} \mathrm{d}\tau'\; g_i(\tau'). \tag{39}$$

The simple interpretation of $G_i(\tau)$ is that it is the probability that a chord length in material i will exceed the length τ. We next define $\tilde{G}_i(\sigma_i)$ as

$$\tilde{G}_i(\sigma_i) = \int_{0}^{\infty} \mathrm{d}\tau\; e^{-\sigma_i\tau}\,G_i(\tau). \tag{40}$$

This is just the Laplace transform of $G_i(\tau)$ evaluated at the transform variable equal to σ_i. In the terms of the $\tilde{G}_i$, we introduce the factor q given by

$$q = \frac{1}{\sigma_0}\left[\frac{1}{\tilde{G}_0(\sigma_0)} - \frac{1}{\lambda_0}\right] + \frac{1}{\sigma_1}\left[\frac{1}{\tilde{G}(\sigma_1)} - \frac{1}{\lambda_1}\right] - 1. \tag{41}$$

According to Levermore et al.,[16] the Markovian two-equation models given by Eqs. (12) and (13), and Eqs. (29) and (30), will then constitute reasonable models for non-Markovian renewal statistics defined by the distribution $g_i(\tau)$ if the replacement

$$\lambda_c \to q\,\lambda_c \tag{42}$$

is made. For (homogeneous) Markovian statistics we have[7]

$$g_i(\tau) = \frac{1}{\lambda_i}\,e^{-\tau/\lambda_i}, \tag{43}$$

which yields

$$G_i(\tau) = e^{-\tau/\lambda_i}, \tag{44}$$

and

$$\tilde{G}_i(\sigma_i) = \lambda_i/(1 + \sigma_i\,\lambda_i). \tag{45}$$

In this case, Eq. (41) gives $q = 1$, and the general non-Markovian model properly reverts to the Markovian model. For inhomogeneous renewal statistics, i.e., if the $g_i(\tau)$ are spatially dependent, this correction factor q would be spatially dependent, but conceptually this causes no difficulties. This simply adds an additional spatial dependence to λ_c that already is allowed an arbitrary dependence.

3. THE NEARLY TRANSPARENT LIMIT

We take as our starting point the two-equation description given by Eqs. (29) and (30). These are the transport-corrected stochastic model equations. We assume that one of the materials, say material zero, is present in small quantities ($p_0 \ll 1$), and that this material is relatively opaque, in that σ_{s0} and σ_{a0} are large compared to σ_{s1} and σ_{a1}. We quantify this by introducing the scalings

$$p_0 \to \varepsilon^2 p_0, \quad \sigma_{s0} \to \sigma_{s0}/\varepsilon^2, \quad \sigma_{a0} = \sigma_{a0}/\varepsilon^2, \tag{46}$$

where ε is the formal smallness parameter, to be set to unity at the end of our considerations. The corresponding material one quantities are taken as $O(1)$. From these scalings, we deduce from Eq. (7) that λ_0 scales as ε^2, and Eqs. (3) and (24) allow us to conclude that S_0 and σ_0 scale as $1/\varepsilon^2$. The quantities λ_1, S_1, and σ_1 are all $O(1)$. Finally, the above scalings lead to the results that $\langle S\rangle$, $\langle\sigma_s\rangle$, and $\langle\sigma\rangle$ are $O(1)$ quantities; λ_c scales as ε^2; T, ν_s, and ν scale as $1/\varepsilon$, and $\hat{\sigma}_s$, and $\hat{\sigma}$ scale as $1/\varepsilon^2$. Introducing these scalings into Eqs. (29) and (30), we find

$$[\mathbf{\Omega}\cdot\nabla + \langle\sigma\rangle]\,\langle I(\mathbf{\Omega})\rangle + \frac{\nu}{\varepsilon}\,\chi(\mathbf{\Omega}) =$$

$$= \langle S\rangle + \frac{1}{4\pi}\int\limits_{4\pi} d\mathbf{\Omega}'\Big[\langle\sigma_s\rangle\,\langle I(\mathbf{\Omega}')\rangle + \frac{\nu_s}{\varepsilon}\,\chi(\mathbf{\Omega}')\Big], \tag{47}$$

$$\left[\mathbf{\Omega}\cdot\nabla + \frac{\hat{\sigma}(\mathbf{\Omega})}{\varepsilon^2}\right]\chi(\mathbf{\Omega}) + \frac{\nu}{\varepsilon}\,\langle I(\mathbf{\Omega})\rangle =$$

$$= \frac{T}{\varepsilon} + \frac{1}{4\pi}\int\limits_{4\pi} d\mathbf{\Omega}'\left[\frac{\hat{\sigma}_s}{\varepsilon^2}\,\chi(\mathbf{\Omega}') + \frac{\nu_s}{\varepsilon}\langle I(\mathbf{\Omega}')\rangle\right]. \tag{48}$$

As ε becomes vanishingly small, these equations describe a small amount of opaque material admixed with a large amount of relatively transparent material. Thus, the stochastic mixture will be nearly transparent, with large and numerous transmission windows between sparse chunks of opaque material.

We now assume asymptotic expansions according to

$$\langle I(\mathbf{\Omega})\rangle \sim \sum_{n=0} \varepsilon^n\,\langle I^{(n)}(\mathbf{\Omega})\rangle, \tag{49}$$

$$\chi(\mathbf{\Omega}) \sim \sum_{n=0} \varepsilon^n\,\chi^{(n)}(\mathbf{\Omega}). \tag{50}$$

We insert Eqs. (49) and (50) into Eqs. (47) and (48), and equate coefficients of like powers of ε. This generates an infinite hierarchy of equations. The first three equations from Eq. (47) are

$$\nu\chi^{(0)}(\mathbf{\Omega}) = \frac{\nu_s}{4\pi}\int\limits_{4\pi} d\mathbf{\Omega}'\,\chi^{(0)}(\mathbf{\Omega}'), \tag{51}$$

$$[\mathbf{\Omega}\cdot\nabla + \langle\sigma\rangle]\,\langle I^{(0)}(\mathbf{\Omega})\rangle + \nu\,\chi^{(1)}(\mathbf{\Omega}) =$$

$$= \langle S\rangle + \frac{1}{4\pi}\int\limits_{4\pi} d\mathbf{\Omega}'\Big[\langle\sigma_s\rangle\,\langle I^{(0)}(\mathbf{\Omega}')\rangle + \nu_s\,\chi^{(1)}(\mathbf{\Omega}')\Big], \tag{52}$$

$$[\mathbf{\Omega}\cdot\nabla + \langle\sigma\rangle]\,\langle I^{(1)}(\mathbf{\Omega})\rangle + \nu\chi^{(2)}(\mathbf{\Omega}) =$$

$$= \frac{1}{4\pi}\int\limits_{4\pi} d\mathbf{\Omega}'\Big[\langle\sigma_s\rangle\,\langle I^{(1)}(\mathbf{\Omega}')\rangle + \nu_s\,\chi^{(2)}(\mathbf{\Omega}')\Big], \tag{53}$$

and the first three equations from Eq. (48) are

$$\hat{\sigma}(\mathbf{\Omega})\,\chi^{(0)}(\mathbf{\Omega}) = \frac{\hat{\sigma}_s}{4\pi}\int\limits_{4\pi} d\mathbf{\Omega}'\chi^{(0)}(\mathbf{\Omega}'), \tag{54}$$

$$\hat{\sigma}(\mathbf{\Omega})\,\chi^{(1)}(\mathbf{\Omega}) + \nu\,\langle I^{(0)}(\mathbf{\Omega})\rangle =$$

$$= T + \frac{1}{4\pi}\int\limits_{4\pi} d\mathbf{\Omega}'\Big[\hat{\sigma}_s\,\chi^{(1)}(\mathbf{\Omega}') + \nu_s\,\langle I^{(0)}(\mathbf{\Omega}')\rangle\Big], \tag{55}$$

$$\mathbf{\Omega}\cdot\nabla\,\chi^{(0)}(\mathbf{\Omega}) + \hat{\sigma}(\mathbf{\Omega})\,\chi^{(2)}(\mathbf{\Omega}) + \nu\,\langle I^{(1)}(\mathbf{\Omega})\rangle =$$

$$= \frac{1}{4\pi}\int\limits_{4\pi} d\mathbf{\Omega}'\,\Big[\hat{\sigma}_s\,\chi^{(2)}(\mathbf{\Omega}') + \nu_s\,\langle I^{(1)}(\mathbf{\Omega}')\rangle\Big]. \tag{56}$$

From Eqs. (51) and (54) we immediately have

$$\chi^{(0)}(\mathbf{\Omega}) = 0. \tag{57}$$

We next multiply Eq. (53) by ε and add the result to Eq. (52). Making use of [see Eq. (49)]

$$\langle I(\mathbf{\Omega})\rangle = \langle I^{(0)}(\mathbf{\Omega})\rangle + \varepsilon\,\langle I^{(1)}(\mathbf{\Omega})\rangle + O(\varepsilon^2), \tag{58}$$

we obtain

$$\mathbf{\Omega}\cdot\nabla\langle I(\mathbf{\Omega})\rangle + \langle\sigma\rangle\,\langle I(\mathbf{\Omega})\rangle + \nu\,\psi(\mathbf{\Omega}) =$$

$$= \langle S\rangle + \frac{\langle\sigma_s\rangle}{4\pi}\int\limits_{4\pi} d\mathbf{\Omega}'\,\langle I(\mathbf{\Omega}')\rangle + \frac{\nu_s\,\eta}{4\pi} + O(\varepsilon^2), \tag{59}$$

where we have defined

$$\psi(\mathbf{\Omega}) = \chi^{(1)}(\mathbf{\Omega}) + \varepsilon\,\chi^{(2)}(\mathbf{\Omega}), \tag{60}$$

$$\eta = \int\limits_{4\pi} d\mathbf{\Omega}\,\psi(\mathbf{\Omega}). \tag{61}$$

Similarly, we multiply Eq. (56) by ε and add the result to Eq. (55) to obtain

$$\hat{\sigma}(\mathbf{\Omega})\,\psi(\mathbf{\Omega}) + \nu\,\langle I(\mathbf{\Omega})\rangle = T + \frac{\nu_s}{4\pi}\int\limits_{4\pi} d\mathbf{\Omega}'\,\langle I(\mathbf{\Omega}')\rangle +$$

$$+ \frac{\hat{\sigma}_s}{4\pi}\,\eta + O(\varepsilon^2). \tag{62}$$

Our procedure will be to solve Eq. (62) for $\psi(\mathbf{\Omega})$ and η, and use those results in Eq. (59).

To this end, we divide Eq. (62) by $\hat{\sigma}(\mathbf{\Omega})$ and integrate over the solid angle. Solving the resulting equation for η gives

$$\eta = \frac{4\pi T}{\hat{\sigma} - \hat{\sigma}_s} - \left(\frac{1}{\bar{\sigma} - \hat{\sigma}_s}\right) \int_{4\pi} d\Omega' \left[\nu \frac{\bar{\sigma}}{\hat{\sigma}(\Omega')} - \nu_s \right] \langle I(\Omega') \rangle + O(\varepsilon^2), \tag{63}$$

where we have defined

$$\frac{1}{\bar{\sigma}} = \frac{1}{4\pi} \int_{4\pi} d\Omega \, \frac{1}{\hat{\sigma}(\Omega)} . \tag{64}$$

Substituting Eq. (63) for η back into Eq. (62) gives $\psi(\Omega)$ as

$$\psi(\Omega) = \frac{\bar{\sigma} T}{(\bar{\sigma} - \hat{\sigma}_s)\hat{\sigma}(\Omega)} - \frac{\nu \langle I(\Omega) \rangle}{\hat{\sigma}(\Omega)} -$$

$$- \frac{\bar{\sigma}}{4\pi(\bar{\sigma} - \hat{\sigma}_s)\, \hat{\sigma}(\Omega)} \int_{4\pi} d\Omega' \left[\frac{\hat{\sigma}_s \nu}{\hat{\sigma}(\Omega')} - \nu_s \right] \langle I(\Omega') \rangle. \tag{65}$$

Finally, the use of Eqs. (63) and (65) for the η and $\psi(\Omega)$ terms in Eq. (59) yields a renormalized equation for the ensemble-averaged specific intensity of radiation. This can be written as

$$\Omega \cdot \nabla \langle I(\Omega) \rangle + \sigma_{\rm eff}(\Omega) \langle I(\Omega) \rangle = S_{\rm eff}(\Omega) +$$

$$+ \frac{1}{4\pi} \int_{4\pi} d\Omega' \sigma_{s,\rm eff}(\Omega', \Omega) \langle I(\Omega') \rangle + O(\varepsilon^2), \tag{66}$$

where we have defined the effective properties according to

$$\sigma_{\rm eff}(\Omega) = \langle \sigma \rangle - \nu^2 / \hat{\sigma}(\Omega) \geq 0, \tag{67}$$

$$\sigma_{s,\rm eff}(\Omega', \Omega) = \langle \sigma_s \rangle - \left(\frac{1}{\bar{\sigma} - \hat{\sigma}_s}\right) \times$$

$$\times \left[\nu_s \nu \bar{\sigma} \left(\frac{1}{\hat{\sigma}(\Omega')} + \frac{1}{\hat{\sigma}(\Omega)} \right) - \frac{\nu^2 \bar{\sigma} \hat{\sigma}_s}{\hat{\sigma}(\Omega') \hat{\sigma}(\Omega)} - \nu_s^2 \right] \geq 0, \tag{68}$$

$$S_{\rm eff}(\Omega) = \langle S \rangle - \left(\frac{1}{\bar{\sigma} - \hat{\sigma}_s}\right) \left[\frac{\bar{\sigma}}{\hat{\sigma}(\Omega)} \nu - \nu_s \right] T \geq 0. \tag{69}$$

As indicated by the inequalities in Eqs. (67) through (69), these effective property definitions are robust in that they always yield nonnegative results for all physical (nonnegative) parameters σ_{si}, σ_{ai}, S_i, and $\lambda_i(\Omega)$, even if one is far from the asymptotic limit under consideration.

We see from Eq. (66) that $\langle I(\Omega) \rangle$ for this nearly transparent asymptotic limit is described by a single deterministic equation of transfer, but with the unusual feature that the effective properties are direction dependent. These angular dependences arise solely from the angular dependences of the correlation length $\lambda_c(\Omega)$, which in its turn arises from the angular dependences of the Markov transition lengths $\lambda_i(\Omega)$. In the case of isotropic statistics, the λ_i are, by definition, independent of Ω and thus $\hat{\sigma}$ is also independent of Ω [see Eq. (32)]. Further, in this case $\bar{\sigma} = \hat{\sigma}$ [see Eq. (64)], and then Eqs. (67) through (69) reduce to angularly independent results given by

$$\sigma_{\rm eff} = \langle \sigma \rangle - \nu^2 / \hat{\sigma} , \tag{70}$$

$$\sigma_{s,\rm eff} = \langle \sigma_s \rangle - \left[\frac{\nu^2}{\hat{\sigma}} - \frac{(\nu - \nu_s)^2}{(\hat{\sigma} - \hat{\sigma}_s)} \right], \tag{71}$$

$$S_{\rm eff} = \langle S \rangle - \left[\frac{(\nu - \nu_s)}{(\hat{\sigma} - \hat{\sigma}_s)} \right] T, \tag{72}$$

which agree with earlier results given by Malvagi et al.[17] We also note that in the limit of a vanishing small correlation length, i.e., $\lambda_c \to 0$, we have $\hat{\sigma} \to \infty$ and then $\bar{\sigma} \to \infty$. Thus, as $\lambda_c \to 0$, Eqs. (67) through (69), as well as Eqs. (70) through (72), reduce to

$$\sigma_{\rm eff} = \langle \sigma \rangle, \quad \sigma_{s,\rm eff} = \langle \sigma_s \rangle, \quad S_{\rm eff} = \langle S \rangle. \tag{73}$$

Thus, this $\lambda_c \to 0$ limit of the nearly transparent asymptotic limit yields the atomic mix model. This is the physically correct result since a vanishing λ_c implies that one or both of the λ_i vanish, and this is the condition for atomic mix to be a valid description of the stochastic problem.

Had we applied the transparent correction procedure, which is what was invoked to replace Eq. (1) with Eq. (23), only to the statistical correction to atomic mix, the renormalized equation of transfer would have been

$$\Omega \cdot \nabla \langle I(\Omega) \rangle + \sigma'_{\rm eff}(\Omega) \langle I(\Omega) \rangle =$$

$$= L_{\rm eff}\, I(\Omega) + S_{\rm eff}(\Omega) + O(\varepsilon^2), \tag{74}$$

with $S_{\rm eff}(\Omega)$ still given by Eq. (69), and

$$\sigma'_{\rm eff}(\Omega) = \sigma_{\rm eff}(\Omega) + \langle \sigma'_s \bar{\mu} \rangle, \tag{75}$$

$$L_{\rm eff} \langle I(\Omega) \rangle = \int_{4\pi} d\Omega' \, \sigma'_{s,\rm eff}(\Omega' \to \Omega) \langle I(\Omega') \rangle, \tag{76}$$

where

$$\sigma'_{s,\rm eff}(\Omega' \to \Omega) = \frac{1}{4\pi} \sigma_{s,\rm eff}(\Omega', \Omega) + \langle \sigma'_s f(\Omega', \Omega) - \frac{\sigma_s}{4\pi} \rangle. \tag{77}$$

In the $\lambda_c \to 0$ limit, Eq. (74) together with the definitions given by Eqs. (69) and (75) through (77), reduces to the atomic mix description for the stochastic equation given by Eq. (1).

Finally, we remark that had we applied the nearly transparent limit scalings to the two-equation stochastic model describing anisotropic scattering as given by Eqs. (12) and (13), we would not have obtained a renormalized equation. The equation in this analysis for

$\psi(\Omega)$ (the analog of Eq. (62)) cannot be solved in terms of $\langle I(\Omega)\rangle$ in any simple closed form. That is, the analog of Eq. (65) cannot be obtained, except in an abstract way involving an inverse scattering operator. This prevents the construction of an explicit renormalized equation. In the next section, we consider a second asymptotic limit, namely the small correlation length limit. In contrast to the nearly transparent limit we have just treated, this small correlation length limit analysis leads to an explicit renormalized equation of transfer from both Eqs. (12) and (13), and Eqs. (29) and (30).

4. THE SMALL CORRELATION LENGTH LIMIT

We again begin our considerations with the isotropic scattering equations given by Eqs. (29) and (30). We assume that the correlation length $\lambda_c(\Omega)$ is small compared to the photon mean free path in each component (clouds and clear sky) of the atmosphere. We reflect this smallness by introducing the scaling

$$\lambda_c(\Omega) \to \varepsilon\, \lambda_c(\Omega), \tag{78}$$

where ε again is a formal smallness parameter. From Eqs. (17), (18), and (31) through (36), we conclude that $\hat{\sigma}(\Omega)$ then scales as $1/\varepsilon$, and all other parameters defined by these equations are $O(1)$. Thus, Eqs. (29) and (30) scale as

$$[\Omega\cdot\nabla + \langle\sigma\rangle]\, \langle I(\Omega)\rangle + \nu\, \chi(\Omega) = \langle S\rangle + \frac{1}{4\pi}\int_{4\pi} d\Omega' \big[\langle\sigma_s\rangle\, \langle I(\Omega')\rangle + \nu_s\, \chi(\Omega')\big], \tag{79}$$

$$\left[\Omega\cdot\nabla + \frac{\hat{\sigma}(\Omega)}{\varepsilon}\right] \chi(\Omega) + \nu\, \langle I(\Omega)\rangle = T + \frac{1}{4\pi}\int_{4\pi} d\Omega' \big[\hat{\sigma}_s\, \chi(\Omega') + \nu_s\, \langle I(\Omega')\rangle\big]. \tag{80}$$

As ε approaches zero, these two equations model a binary stochastic mixture with a vanishingly small correlation length, which in turn implies that one or both of the $\lambda_i(\Omega)$ is vanishingly small.

We introduce the asymptotic expansions given by Eqs. (49) and (50) into Eqs. (79) and (80), and equate coefficients of like powers of ε. The first two equations from Eq. (79) are

$$[\Omega\cdot\nabla + \langle\sigma\rangle]\, \langle I^{(0)}(\Omega)\rangle + \nu\, \chi^{(0)}(\Omega) = \langle S\rangle + \frac{1}{4\pi}\int_{4\pi} d\Omega' \big[\langle\sigma_s\rangle\, \langle I^{(0)}(\Omega')\rangle + \nu_s\, \chi^{(0)}(\Omega')\big], \tag{81}$$

$$[\Omega\cdot\nabla + \langle\sigma\rangle]\, \langle I^{(1)}(\Omega)\rangle + \nu\, \chi^{(1)}(\Omega) = \frac{1}{4\pi}\int_{4\pi} d\Omega' \big[\langle\sigma_s\rangle\, \langle I^{(1)}(\Omega')\rangle + \nu_s\, \chi^{(1)}(\Omega')\big], \tag{82}$$

and the first two equations from Eq. (80) are

$$\hat{\sigma}(\Omega)\, \chi^{(0)}(\Omega) = 0, \tag{83}$$

$$\Omega\cdot\nabla\, \chi^{(0)}(\Omega) + \hat{\sigma}(\Omega)\, \chi^{(1)}(\Omega) + \nu\, \langle I^{(0)}(\Omega)\rangle = T + \frac{1}{4\pi}\int_{4\pi} d\Omega' \big[\hat{\sigma}_s\, \chi^{(0)}(\Omega') + \nu_s\, \langle I^{(0)}(\Omega')\rangle\big]. \tag{84}$$

Since $\hat{\sigma}(\Omega) > 0$, we deduce from Eq. (83) that

$$\chi^{(0)}(\Omega) = 0, \tag{85}$$

and then Eq. (81) becomes

$$[\Omega\cdot\nabla + \langle\sigma\rangle]\langle I^{(0)}(\Omega)\rangle = \langle S\rangle + \frac{\langle\sigma_s\rangle}{4\pi}\int_{4\pi} d\Omega' \langle I^{(0)}(\Omega')\rangle. \tag{86}$$

Making use of Eq. (49) in the form

$$\langle I(\Omega)\rangle = \langle I^{(0)}(\Omega)\rangle + O(\varepsilon), \tag{87}$$

the result can be rewritten as

$$\Omega\cdot\nabla\, \langle I(\Omega)\rangle + \langle\sigma\rangle\, \langle I(\Omega)\rangle = \langle S\rangle + \frac{\langle\sigma_s\rangle}{4\pi}\int_{4\pi} d\Omega'\, \langle I(\Omega')\rangle + O(\varepsilon). \tag{88}$$

From Eq. (88) we deduce that, with an error of $O(\varepsilon)$, i.e., with an error of $O(\lambda_c)$, the proper statistical description for a small correlation length problem is simply atomic mix. This result is, of course, known from physical considerations.

To obtain the first order (in λ_c) correction to atomic mix, we need to utilize Eqs. (82) and (84). We first multiply Eq. (82) by ε and add the result to Eq. (81). Using Eq. (58) and defining

$$\psi(\Omega) = \varepsilon\, \chi^{(1)}(\Omega), \tag{89}$$

we obtain

$$\Omega\cdot\nabla\langle I(\Omega)\rangle + \langle\sigma\rangle\, \langle I(\Omega)\rangle + \nu\, \psi(\Omega) = \langle S\rangle + \frac{1}{4\pi}\int_{4\pi} d\Omega' [\langle\sigma_s\rangle\, \langle I(\Omega')\rangle + \nu_s\, \psi(\Omega') + O(\varepsilon^2). \tag{90}$$

Next we multiply Eq. (84) with $\chi^{(0)}(\Omega)=0$ by ε, make use of the definition of $\psi(\Omega)$ given by Eq. (89), and use [see Eq. (49)]

$$\varepsilon\langle I^0(\Omega)\rangle = \varepsilon\langle I(\Omega)\rangle + O(\varepsilon^2). \tag{91}$$

This gives, setting the formal smallness parameter ε to unity,

$$\hat{\sigma}(\Omega)\, \psi(\Omega) + \nu\, \langle I(\Omega)\rangle = T + \frac{\nu_s}{4\pi}\int_{4\pi} d\Omega'\, \langle I(\Omega')\rangle + O(\varepsilon^2). \tag{92}$$

Using $\psi(\boldsymbol{\Omega})$ from this equation in Eq. (90) yields, upon collecting terms,

$$\boldsymbol{\Omega}\cdot\nabla\langle I(\boldsymbol{\Omega})\rangle + \left[\langle\sigma\rangle - \frac{\nu^2}{\hat{\sigma}(\boldsymbol{\Omega})}\right]\langle I(\boldsymbol{\Omega})\rangle =$$

$$= \langle S\rangle - \left[\frac{\nu}{\hat{\sigma}(\boldsymbol{\Omega})} - \frac{\nu_s}{\bar{\sigma}}\right] T +$$

$$+ \frac{1}{4\pi}\int_{4\pi} d\boldsymbol{\Omega}' \left\{\langle\sigma_s\rangle - \left[\frac{\nu\,\nu_s}{\hat{\sigma}(\boldsymbol{\Omega}')} + \frac{\nu\,\nu_s}{\hat{\sigma}(\boldsymbol{\Omega})} - \frac{\nu_s^2}{\bar{\sigma}}\right]\right\} \times$$

$$\times \langle I(\boldsymbol{\Omega}')\rangle + O(\varepsilon^2), \tag{93}$$

where $\bar{\sigma}$ has been defined earlier by Eq. (64). This result can be written as

$$\boldsymbol{\Omega}\cdot\nabla\langle I(\boldsymbol{\Omega})\rangle + \sigma_{\text{eff}}(\boldsymbol{\Omega})\,\langle I(\boldsymbol{\Omega})\rangle = S_{\text{eff}}(\boldsymbol{\Omega}) +$$

$$+ \frac{1}{4\pi}\int_{4\pi} d\boldsymbol{\Omega}'\sigma_{s,\text{eff}}(\boldsymbol{\Omega}', \boldsymbol{\Omega})\,\langle I(\boldsymbol{\Omega}')\rangle + O(\varepsilon^2), \tag{94}$$

where we have defined the effective parameters

$$\sigma_{\text{eff}}(\boldsymbol{\Omega}) = \langle\sigma\rangle - \nu^2/\hat{\sigma}(\boldsymbol{\Omega}), \tag{95}$$

$$\sigma_{s,\text{eff}}(\boldsymbol{\Omega}', \boldsymbol{\Omega}) = \langle\sigma_s\rangle - \nu_s\left\{\nu\left[\frac{1}{\hat{\sigma}(\boldsymbol{\Omega}')} + \frac{1}{\hat{\sigma}(\boldsymbol{\Omega})}\right] - \frac{\nu_s}{\bar{\sigma}}\right\}, \tag{96}$$

$$S_{\text{eff}}(\boldsymbol{\Omega}) = \langle S\rangle - \left[\frac{\nu}{\hat{\sigma}(\boldsymbol{\Omega})} - \frac{\nu_s}{\bar{\sigma}}\right] T. \tag{97}$$

From Eq. (32) we have

$$1/[\hat{\sigma}(\boldsymbol{\Omega})] = \lambda_c(\boldsymbol{\Omega}) + O(\lambda_c^2), \tag{98}$$

and then Eq. (64) yields

$$1/\bar{\sigma} = \bar{\lambda}_c + O(\lambda_c^2), \tag{99}$$

where we have defined

$$\bar{\lambda}_c = \frac{1}{4\pi}\int_{4\pi} d\boldsymbol{\Omega}\,\lambda_c(\boldsymbol{\Omega}). \tag{100}$$

Thus, Eqs. (95) through (97) can be written, since $O(\lambda_c^2) = O(\varepsilon^2)$,

$$\sigma_{\text{eff}}(\boldsymbol{\Omega}) = \langle\sigma\rangle - \nu^2\,\lambda_c(\boldsymbol{\Omega}) + O(\varepsilon^2), \tag{101}$$

$$\sigma_{s,\text{eff}}(\boldsymbol{\Omega}', \boldsymbol{\Omega}) = \langle\sigma_s\rangle - \nu_s\,\{\nu[\lambda_c(\boldsymbol{\Omega}') + \lambda_c(\boldsymbol{\Omega})] -$$

$$- \nu_s\,\bar{\lambda}_c\} + O(\varepsilon^2), \tag{102}$$

$$S_{\text{eff}}(\boldsymbol{\Omega}) = \langle S\rangle - [\nu\,\lambda_c(\boldsymbol{\Omega}) - \nu_s\,\bar{\lambda}_c]\,T + O(\varepsilon^2). \tag{103}$$

The $O(\varepsilon^2)$ errors in Eqs. (101) through (103) are consistent with the $O(\varepsilon^2)$ error in the renormalized equation of transfer given by Eq. (94).

Another asymptotically equivalent, but more robust, set of these effective properties is

$$\sigma_{\text{eff}}(\boldsymbol{\Omega}) = \frac{\langle\sigma\rangle}{1 + \nu^2\,\lambda_c(\boldsymbol{\Omega})/\langle\sigma\rangle} + O(\varepsilon^2), \tag{104}$$

$$\sigma_{s,\text{eff}}(\boldsymbol{\Omega}', \boldsymbol{\Omega}) =$$

$$= \frac{\langle\sigma_s\rangle}{1 + \nu_s\,\{\nu\,[\lambda_c(\boldsymbol{\Omega}') + \lambda_c(\boldsymbol{\Omega})] - \nu_s\,\bar{\lambda}_c\}/\langle\sigma_s\rangle} + O(\varepsilon^2), \tag{105}$$

$$S_{\text{eff}}(\boldsymbol{\Omega}) = \frac{\langle S\rangle}{1 + [\nu\,\lambda_c(\boldsymbol{\Omega}) - \nu_s\,\bar{\lambda}]\,T/\langle S\rangle} + O(\varepsilon^2). \tag{106}$$

For isotropic statistics, i.e., λ_c independent of $\boldsymbol{\Omega}$, Eqs. (101) through (103) further reduce to

$$\sigma_{\text{eff}}(\boldsymbol{\Omega}) = \langle\sigma\rangle - \nu^2\,\lambda_c + O(\varepsilon^2), \tag{107}$$

$$\sigma_{s,\text{eff}} = \langle\sigma_s\rangle - \nu_s\,(2\nu - \nu_s)\lambda_c + O(\varepsilon^2), \tag{108}$$

$$S_{\text{eff}} = \langle S\rangle - (\nu - \nu_s)\,\lambda_c + O(\varepsilon^2), \tag{109}$$

with a similar simplification for Eqs. (104) through (106).

This same asymptotic limit, namely the small correlation length limit, but now corresponding to the anisotropic scattering model given by Eqs. (12) and (13), is easily treated just as we have treated the isotropic scattering model given by Eqs. (29) and (30). All terms in Eqs. (12) and (13) are scaled as $O(1)$, except $\hat{\sigma}'(\boldsymbol{\Omega})$ which scales as $1/\varepsilon$. Once again, we use the expansions given by Eqs. (49) and (50), and equate coefficients of like powers of ε. Omitting the straightforward details, which closely parallel the analysis just completed, we find a first order in $\lambda_c(\boldsymbol{\Omega})$ correction to atomic mix given by

$$\boldsymbol{\Omega}\cdot\nabla\langle I(\boldsymbol{\Omega})\rangle + \sigma'_{\text{eff}}(\boldsymbol{\Omega})\,\langle I(\boldsymbol{\Omega})\rangle =$$

$$= S_{\text{eff}}(\boldsymbol{\Omega}) + L_{\text{eff}}\,\langle I(\boldsymbol{\Omega})\rangle + O(\varepsilon^2). \tag{110}$$

Here we have defined the effective properties

$$\sigma'_{\text{eff}}(\boldsymbol{\Omega}) = \langle\sigma'\rangle - \nu'^2/\hat{\sigma}'(\boldsymbol{\Omega})6 \tag{111}$$

$$S_{\text{eff}}(\boldsymbol{\Omega}) = \langle S\rangle - \left[\frac{\nu'}{\hat{\sigma}'(\boldsymbol{\Omega})} - K\,\frac{1}{\hat{\sigma}'(\boldsymbol{\Omega})}\right] T, \tag{112}$$

and the effective scattering operator

$$L_{\text{eff}} = \langle L\rangle - \nu'\left[\frac{1}{\hat{\sigma}'(\boldsymbol{\Omega})}K + K\,\frac{1}{\hat{\sigma}'(\boldsymbol{\Omega})}\right] + K\,\frac{1}{\hat{\sigma}'(\boldsymbol{\Omega})}K. \tag{113}$$

Again we see that the effective properties contain unusual angular dependences arising from the angular dependence of $\lambda_c(\mathbf{\Omega})$. Further, the effective source as given by Eq. (112) involves the scattering operator K, and the effective scattering operator defined by Eq. (113) is somewhat complex in that it involves a convolution of the scattering operator K with itself. Nonetheless, this analysis yields a single, renormalized equation of transfer given by Eq. (110). That is, the two-equation model given by Eqs. (12) and (13) has been reduced to a single equation.

As before, making use of Eq. (98), the effective quantities given by Eqs. (111) through (113) can be written as atomic mix quantities plus a term linear in $\lambda_c(\mathbf{\Omega})$. These results are

$$\sigma'_{\text{eff}}(\mathbf{\Omega}) = \langle\sigma'\rangle - \nu'^2\,\lambda_c(\mathbf{\Omega}) + O(\varepsilon^2), \quad (114)$$

$$S_{\text{eff}}(\mathbf{\Omega}) = \langle S\rangle - (\nu' - K)\,\lambda_c(\mathbf{\Omega})\,T + O(\varepsilon^2), \quad (115)$$

$$L_{\text{eff}} = \langle L\rangle + \nu'[\lambda_c(\mathbf{\Omega})K + K\,\lambda_c(\mathbf{\Omega})] + K\,\lambda_c(\mathbf{\Omega})K + O(\varepsilon^2). \quad (116)$$

A more robust form of Eq. (114) is

$$\sigma'_{\text{eff}}(\mathbf{\Omega}) = \frac{\langle\sigma'\rangle}{1 + [\nu'^2\,\lambda_c(\mathbf{\Omega})/\langle\sigma'\rangle]} + O(\varepsilon^2). \quad (117)$$

Similar simple manipulations of Eqs. (115) and (116) are not possible because of the appearance of the scattering operators K and $\langle L\rangle$. That is, these manipulations would result in formal inverse scattering operators present in the results.

5. CONCLUDING REMARKS

In this paper we have shown that a two-equation model[7–11] describing radiative transfer in a binary (clouds and clear sky) Markovian mixture can be reduced to a single renormalized equation of transfer in two distinct circumstances. These two circumstances are asymptotic limits corresponding to: (1) a nearly transparent atmosphere and (2) small correlation length mixing statistics. We also pointed out that these Markovian models can be applied to non-Markovian statistics of the renewal type by appropriately modifying the correlation length.[16] This is important in the cloud-radiation interaction problem because, while the spacing between clouds is perhaps well approximated by Markovian statistics, the clouds themselves are almost certainly not Markovian in nature.[18]

Before any of these renormalized equations are incorporated into general circulation models of the atmosphere, numerical tests should be performed to assess their accuracy away from the asymptotic limits underlying their derivations. Specifically, renormalized equation results should be compared to the corresponding two-equation model predictions. In turn, both of these results should be compared to exact stochastic atmosphere results, computed by: (1) constructing, by Monte Carlo methods, a statistical realization of a partially cloudy atmosphere[9,19]; (2) solving the resulting deterministic radiative transfer problem either by Monte Carlo or by a deterministic method such as discrete ordinates[9,15,19]; and, (3) repeating the above two steps a large number of times, of the order of 10^5 (Refs. 9 and 19) and numerically averaging to obtain the ensemble-averaged solution. A comparison of higher moments, such as the variance, might also be of some interest. In this regard, the Liouville master equation derivation of the two-equation stochastic model for the ensemble-averaged intensity can also be used to obtain corresponding two-equation models for all of the higher moments, and in particular for the variance.[7] Finally, since two-stream (diffusive) approximations[20] are often used in practice (because of their simplicity) to treat atmospheric radiative transfer problems, there would be interest in a numerical comparison of two-stream renormalized equation results with full angularly dependent renormalized equation predictions.

In summary, a great deal of numerical testing is clearly required to establish the regions of validity of the various renormalized equations of transfer (and their diffusive approximations) proposed in this paper. If those regions of validity coincide with the regions of interest in the cloud-radiation arena, it would seem that these renormalized equations would be good candidates for inclusion as the radiative transfer treatment in general circulation models of the atmosphere.

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