Causality and demodulation of optical monotone-phase signals

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The state-of-the-art in definition of the amplitude and phase of signals as applied to analysis of fringe patterns of the interferogram type is discussed. Two problems of both theoretical and practical significance, namely, the conditions for existence of two-band spectrum in real signals and optimal calculation of the Hilbert transform, are considered. Some constructive results are presented.

Introduction

Signals, depending on two variables, in the form of oscillations, of closed or unclosed fringes are observed in interferometry, when studying surfaces, transparent bodies, and wave processes. They may have different physical origins. Such signals also can be formed in a natural way as, for example, tree rings, which are indicators of changes of numerous climatic factors. Variations of the shape and mutual arrangement of fringes carry the information on the object under study and can be considered as spatial amplitude-phase modulation.

In the general case, to describe an oscillation or wave process by two functions (amplitude and phase), a consistent definition of these two concepts is required. The wave equation or the equation describing some oscillation process does not include such a definition; therefore, some additional reasoning is needed. A great number of papers (Refs. 2–6, 9, 10, 21, 27, 32, 34, and 39) treat the problem of definition of the amplitude and phase as applied to the process depending on one variable. Various ways of definition are useful in framework of different problems to be solved and for mathematical models in use. However, the analytic signal (AS) introduced by Gabor in 1946 (Ref. 27) has received the widest acceptance due to Vakman's works.

Fringe-patterns are the subject of permanent research. The urgency of such research is confirmed, for example, by Fringe-conference held annually in Bremen at the Bremer Institut für Angewandte Strahltechnik.[18] In Tomsk, at the Institute of Atmospheric Optics, such a research is carrying out since the Institute inception. Initially, S.S. Khmelevtsov, V.V. Pokasov, V.P. Lukin, and O.N. Emaleev designed and made a phasometer and conducted first field measurements of phase fluctuations of laser radiation propagating through the atmosphere.

In 1974, I was involved in development of mathematical apparatus for description of interference patterns for their further computer analysis. Then, starting from 1982 in the Joint Institute of Atmospheric Optics SB AS USSR, methods of high-precision interference testing of astronomic optics were developed by me in close cooperation with L.A. Pushnoy and E.A. Vitrichenko (Section headed by Academician A. Prokhorov in the Space Research Institute AS USSR). The methods were based on the formalism of analytic signal and dispersion relations and provided for the root-mean-square error in surface measurement less than one hundredth of wavelength.[7,8] The method of interferogram demodulation by filtering in the trigonometric basis, which is widely used now, was proposed by our team in 1982 and, simultaneously, by Takeda, Ina, and Kobayashi.[38]

With development of adaptive optics, a demand arose for phase modulation of a light wave by changing the shape of an optical surface in space and time. Therefore, it became necessary to define consistently and constructively the light wave phase, what was done through generalization of AS to the spatiotemporal case.[19]

Nowadays these methods are progressed in the Institute of Optical Monitoring. There appeared a possibility to apply them to analysis of tree rings, which are among few sources of information on long-term changes in the temperature, humidity, and chemical composition of soil and the atmosphere. It is also possible to find some dendro-optical analogies and to consider tree rings as an interferogram formed by some "ecological" and "biological" fields.

In this review, we consider the fundamentals of the analytic signal formalism and related problems, present the extension of capabilities of the analytic signal and a departure from it in a special case, and propose an optimal numerical realization of the basic algorithm just the Hilbert transform.

1. Definition of the amplitude and phase

The simplest mathematical model of signals depending on two variables can be of the form

$$G(x, y) = \mathbf{B}\,[\,|1 + W(x, y)|^2] = \\ = \mathbf{B}\,[1 + a^2(x, y) + 2a(x, y)\cos \mathrm{N}(x, y)]. \quad (1)$$

Here $W(x, y)$ is the complex object field with the amplitude $a(x, y) < 1$ and phase $\mathrm{N}(x, y)$; $\mathbf{B}$ is the operator characterizing a nonlinear detector. The

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transformation performed by it may be both reversible and irreversible, for example, binary.

The fundamental property providing for the fringe structure of a signal is monotonicity of the phase $N(x, y)$ in the parametric cross section of the plane xy. In this case, the extrema of the signal defined in the cross section should coincide with the extrema of the signal defined on the whole plane. This property provides for the possibility of signal demodulation in some independent cross sections.[19] Therefore, from here on we use only one argument x.

The cosine function entering into the signal model (1) is periodic and even. Therefore, a non-monotonic phase can give the same signal modulation as the monotonic one. If x_0 is the phase extreme point, then $\cos N(x) = \cos \tilde{N}(x)$, where

$$\tilde{N}(x)=\left\{\begin{array}{l} 2\pi n+N(x),\ x<x_0 \\ 2\pi(n+1)-N(x),\ x>x_0 \\ \pi(2n+1),\ x=x_0 \end{array}\right|_{n\in(-\infty,\infty)} . \quad (2)$$

The phase $\tilde{N}(x)$ can be discontinuous at the point x_0, but its derivative remains continuous and bounded in the absolute value to the same limit as derivative of $N(x)$.

Analyzing different methods for definition of the amplitude and phase, Vakman took into account the continuity and differentiability of the signal amplitude, phase independence of the units, in which the signal is measured, and coincidence with the intuitive ideas of the amplitude and phase of harmonic oscillations. The conclusions have been formulated as follows.

– For harmonic oscillations, the acceptable definitions give the expected result, continuous amplitude, and linear phase.

– For narrow-band signals, the results given by different methods can disagree, but the discrepancy between them decreases as the relative width of the signal spectral band decreases.

– Some methods lead to appearance of singular points in the amplitude and phase, when signal derivatives turn to zero.

– Consistent and most general definition of the amplitude and phase is achieved using the analytic signal.

The analytic signal $W(x)$ is constructed as a complex function

$$W(x) = U(x) + iV(x),$$

$$V(x)=\frac{1}{\pi}\mathrm{v.p.}\int_{-\infty}^{\infty}\frac{U(s)}{x-s}\mathrm{d}s=\mathbf{H}\,U(x) . \quad (3)$$

Here the improper integral is determined in the sense of Caushy's principal value (v.p.) in the cases that $s \to \pm\infty$ and at $x = s$. The imaginary part of the analytical signal $V(x)$ is the Hilbert transform of its real part $U(x)$. The operator $\mathbf{H}$ denotes the Hilbert transform with respect to the argument x. It was shown[3] to be the unique linear operator, for which the following equality is valid:

$$\mathbf{H}\cos(\alpha_c x+\varphi_0)=\sin(\alpha_c x+\varphi_0), \quad (4)$$

where φ_0 and $\alpha_c > 0$ are constants, having the sense of the initial phase and carrier frequency. Thus, within the rule (3), AS is introduced in the same way for all real signals. However, after introduction of the AS, the amplitude and the phase are calculated by the well-known equations with ambiguity in the signs and in the number of π for the phase, therefore the obtained functions are not necessarily unique:

$$a(x)=\pm\sqrt{U^2(x)+V^2(x)},$$
$$N(x)=\arctan\frac{V(x)}{U(x)}\pm n\pi,\ n=0,1,2,\dots . \quad (5)$$

What is interesting, the signs and the integer number n can vary with the argument x (Ref. 6).

The most important property of the analytic signal is the causality of its Fourier transform.[12] The causality follows from the equivalence of the Hilbert transform to multiplication by the function $-i\,\mathrm{sgn}\,\alpha$ in the frequency region α. In other words, AS corresponding to the given real function $U(x)$ is obtained by zeroing a half of its spectrum

$$W(x)=\int_{-\infty}^{\infty}(1+\mathrm{sgn}\,\alpha)\,\mathrm{e}^{-i\alpha x}\,\mathrm{d}\alpha\int_{-\infty}^{\infty}U(s)\,\mathrm{e}^{i\alpha s}\,\mathrm{d}s=$$
$$=2\int_{0}^{\infty}\mathrm{e}^{-i\alpha x}\,\mathrm{d}\alpha\int_{-\infty}^{\infty}U(s)\,\mathrm{e}^{i\alpha s}\,\mathrm{d}s . \quad (6)$$

Suitability of a mathematical model for investigations is largely determined by properties of the functions, it consists of. When selecting a class of functions for representation of a physical quantity, one should take into account the existence of the Hilbert transform, which introduces the analytic signal. The periodic functions are of special interest, because the algorithm of fast Fourier transform (FFT), which is basic in numerical analysis of signals, is intended just for them. In this connection, it is worth approximating experimental data by some periodic function for obtaining the optimal spectral estimates.[16] This problem is considered in Section 3.

The spectra of periodic functions contain singularities in the form of δ-functions. According to the Paley–Wiener–Schwarz theorem,[23] such functions have finite spectra and are entire analytic functions of the exponential type (EFET). What is more, if EFET bounded on the real axis, it also is a function of class A (according to Levin, Ref. 11) or class B (Ref. 33). The boundedness of EFET determines the absolute integrability of its Fourier transform (spectrum), including the case that EFET itself is not square-integrable. The contrary proposition is valid as well, and because the Hilbert transform of a function does not break the absolute integrability of its spectrum, the

Hilbert transform of EFET is a bounded function as well. The Hilbert transform exists also for the functions having no finite spectrum, but with a continuous derivative, as well as for a wider class of functions satisfying the Hölder condition. However, these classes will not be used here because of the need in discrete representation of signals.

The problems of definition of the amplitude and the phase are discussed intensively, but they are open yet.[6,9] The cause is that there is no unique approach to construction of mathematical models of oscillations, waves, detectors, and systems, because the devices and signal carriers have different physical nature. For example, the position of the origin with respect to the signal in space is of no concern, but there always is the initial instant in time, before which nothing was going on. However, the AS gives nonzero amplitude even before this instant. Therefore, some doubts are cast upon the expediency of such a model, which should be justified.

An entire function is completely determined by its zeros, for example, in the form of Hadamard's product. At weak modulation, zeros of the oscillation $W(x)$ lie in the complex half-planes, $|W(x)| \neq 0$. As modulation intensifies, zeros appear on the real axis x, what realizes the ambiguity contained in Eq. (5). Thus, we can say that zeros are indicators of a threshold of phenomenon complexity.[17] During propagation, with the increase of the path and with intensification of medium fluctuations, the light wave gets over the threshold of complexity and vortices are formed in it. At the vortex centers, the amplitude is zero, and the phase is indefinite there. However, in some neighborhood of the vortex center, the phase varies monotonically around zero and is minimal.[20] Besides, the probability density of amplitude fluctuations of the wave is changed, and some approximate methods of solution of the wave equation become inapplicable after the threshold. It follows here from that the complexity of mathematical description of oscillation and wave processes increases considerably after appearance of zeros. In this sense, the analytic signal does not simplify the situation. Nevertheless, application of the AS to analysis of two-dimensional spatial signals is useful.[19] For definiteness, let us formulate the conditions and the order of introduction of the analytic signal.

a) The real signal is formed in such a way that it has a two-band Fourier spectrum; the left-hand and the right-hand spectral bands do not overlap.

b) The associated complex signal is formed by zeroing of one of the spectral bands of the doubled real signal.

Now consider two practically important problems.

2. Phase monotonicity and dispersion causality of spectrum

From the interference pattern $G(x, y)$, one can hardly conclude whether the cross sections $\mathrm{Re}W(x)$ have two-band spatial spectra, what is needed for application of the analytic signal. These cross sections may be a broadband signal, whose carrier frequency differs a little from the halfwidth of the spectral band. In this case, the point of origin may lie inside the spectral band and the possibility to apply the analytic signal to define the amplitude and phase becomes problematic.

Experimentally, we can observe the complete profile of interference fringes in linear cross sections of an interferogram, what is provided for by the phase monotony in this cross section. The monotony can be also assumed in the case of tree rings in view of their non-negative radial growth.

Keeping in mind that the frequency shift of the spectrum is equivalent to addition of a linear function to the signal phase, we can connect the fulfillment of the causality condition for the spectrum of the complex function $W(x)$ with the monotonic character of its phase. Let us use the Bernstein inequality[13,22] in the form

$$\max\left|\frac{\mathrm{d}W(x)}{\mathrm{d}x}\right| \le \alpha_s \max|W(x)|, \tag{7}$$

where the function $W(x) = a(x)\exp i\mathrm{N}(x)$ belongs to the class of functions with the finite spectrum; α_s is the absolute value of the upper frequency in the spectrum of this function.

In an important special case, characteristic of interference testing, that $\mathrm{N}(x)$ is not a linear function and the amplitude $a(x)$ is constant, we find

$$\max|\mathrm{N}'(x)| \le \alpha_s . \tag{8}$$

According to the theorem on the spectral shift, the causality condition is fulfilled for the function $a(x)\exp i\,[\mathrm{N}(x) + \alpha_s x]$ in the frequency region, and it follows from the Bernstein inequality that the phase $\mathrm{N}(x) + \alpha_s x$ is certainly monotonic. Thus, for the phase of signal with the constant amplitude to be monotonic, it is sufficient for the spectrum of the signal to be causal. Obviously, the cases are possible that the phase is monotonic but the causality is absent.

Let us determine the effective width of the spectrum α_e as the normalized second-order moment. Such an approach allows us to consider functions having no finite spectrum as well. Let $W(x)$ be a T-periodic function bounded above. Then the discrete Fourier spectrum S_k exists for it. Keeping in mind Parseval's identity, we can find

$$\alpha_e^2 = \frac{\sum_{k=-\infty}^{\infty} k^2 |S_k|^2}{\sum_{k=-\infty}^{\infty} |S_k|^2} = \frac{\int_0^T |W'(x)|^2 \mathrm{d}x}{\int_0^T |W(x)|^2 \mathrm{d}x} =$$

$$= \frac{\int_0^T [a'^2 + a^2 \mathrm{N}'^2]\mathrm{d}x}{\int_0^T a^2 \mathrm{d}x} < A^2 + \max \mathrm{N}'^2(x), \tag{9}$$

where

$$A^2=\int_0^T a'^2(x)\mathrm{d}x \Big/ \int_0^T a^2(x)\mathrm{d}x \,.$$

By analogy with the causality condition, we introduce the condition of dispersion causality. For this condition to be fulfilled, it is sufficient to shift the spectrum S_k in frequency area more than its effective width α_e in any direction as follows

$$a(x)\exp i\left[\mathrm{N}(x)+x\sqrt{A^2+\max \mathrm{N}'^2(x)}\,\right]. \quad (10)$$

At sufficiently large shift, the phase monotonicity and dispersion causality of the spectrum are observed simultaneously, as it follows from the inequality

$$\alpha_e < \sqrt{A^2+\max \mathrm{N}'^2(x)} \geq \max\left|\mathrm{N}'(x)\right|. \quad (11)$$

At a = const, $A = 0$ and we obtain from Eq. (11)

$$\max\left|\phi'(x)\right| > \alpha_e\,. \quad (12)$$

Thus, for the condition of dispersion causality of the spectrum of a signal with the constant amplitude to be fulfilled, it is sufficient for the signal phase to be monotonic.

Comparing Eqs. (8) and (12), we obtain

$$\alpha_e < \max\left|\mathrm{N}'(x)\right| \leq \alpha_s, \quad (13)$$

whence it follows that the condition of causality is stronger than the condition of dispersion causality, which can consequently find the wider utility for construction of the complex signal by the rules (see Items (**a**) and (**b**) in Section 1).

To characterize the position of the signal spectrum with respect to the origin, it is useful to introduce the degree of causality

$$p_c = \pm\sqrt{\left(\sum_{k=1}^{NN}|S_k|^2 - \sum_{k=NN+1}^{N}|S_k|^2\right)\Big/\sum_{k=1}^{N}|S_k|^2}\,, \quad (14)$$

where $NN = N/2 + 1$ is the Nyquist frequency; S_k is the discrete spectrum of the signal calculated by the FFT algorithm. It is obvious that $|p_c| \leq 1$, and the equality is achieved, when the spectrum of the complex signal satisfies the condition of causality.

The monotonic character of the phase makes it possible to increase the degree of causality of the spectrum. Assume that the phase is monotonic and the function $W(x) = a(x)\exp i\mathrm{N}(x)$ is not an analytic signal. If the variable x is transformed so that

$$\mathrm{N}(x) = \alpha_c\,\tau, \quad x = \mathrm{N}^{-1}(\alpha_c\,\tau), \quad (15)$$

then the spectrum of the function $W(\tau)$ becomes far narrower and concentrated nearby the point $\alpha = \alpha_c$. With neglect of transformation errors, the width of this spectrum with respect to α_c depends only on the amplitude $a(\tau)$, which changes only slightly as compared to $a(x)$. At $a(x)$ = const, the spectrum of the function $W(x)$ satisfies the condition of dispersion causality, and the transformed function $W(\tau)$ is complex harmonic oscillation with the frequency α_c.

The single-valuedness of the inverse function N^{-1} is provided for by the monotonic character of the function $\mathrm{N}(x)$ itself. If the derivative of $\mathrm{N}(x)$ is nonzero as well, then the inverse function has no breaks, what is especially important for numerical realization.

This transformation **Œ** (compression–extension) compresses the periods of oscillation, which are larger than some mean period, and extends those of them, which are smaller than the mean period, while the inverse transformation Œ^{-1} returns oscillations into the initial state according to the following equations:

$$\text{Œ}a(x)\cos\mathrm{N}(x) = a\{\mathrm{N}^{-1}(\alpha_c\tau)\}\cos\{\mathrm{N}[\mathrm{N}^{-1}(\alpha_c\tau)]\} =$$
$$= a(\tau)\cos\alpha_c\tau,$$
$$\mathbf{H}a(\tau)\cos\alpha_c\tau = a(\tau)\mathrm{H}\cos\alpha_c\tau = a(\tau)\sin\alpha_c\tau, \quad (16)$$
$$\text{Œ}^{-1}a(\tau)\sin\alpha_c\tau = a\{\mathrm{N}(x)/\alpha_c\}\sin\{\alpha_c\mathrm{N}(x)/\alpha_c\} =$$
$$= a(x)\sin\mathrm{N}(x).$$

The second equation here is valid under the condition that the amplitude $a(x)$ does not contain frequencies higher than α_c, therefore this function is put before[24] the operator of the Hilbert transform with respect to the variable τ. Figures 1–3 illustrate the above said.

In the general case, similarly to Eq. (3), we introduce the associated complex signal $W(x)$ for the real signal $U(x)$ with the monotonic phase by the rule

$$W(x) = U(x) + iV(x), \quad V(x) = \text{Œ}^{-1}\mathbf{H}\,\text{Œ}U(x). \quad (17)$$

According to the method of determining the width of the spectrum (9), this complex signal $W(x)$ can be called the effective signal. It should be applied in place of the AS at the *a priori* monotonic phase and when the function $U(x)$ has not a two-band spectrum.

The question arises how to implement the operations described above? Actually, in order to determine the phase $\mathrm{N}(x)$ having only $U(x)$, the phase should be pre-known for the transformations (15)–(17) to be performed. However, we can assume here that

$$\mathrm{N}_0(x) = \arctan\left[\mathbf{H}\,U(x)/U(x)\right] \quad (18)$$

is sufficient for the initial compression of the spectrum. Then the iteration process is performed according to the equation

$$\mathrm{N}_{n+1}(x) =$$
$$= \arctan\left\{\mathbf{H}\,U[\mathrm{N}_n^{-1}(\alpha_c\,\tau)]/U[\mathrm{N}_n^{-1}(\alpha_c\,\tau)]\right\}\Big|_{\tau=\mathrm{N}_n(x)/\alpha_c}. \quad (19)$$

Numerical experiments in Ref. 19 have shown that four iterations by Eq. (19) increase the accuracy of phase estimation more than tenfold as compared with Eq. (18).

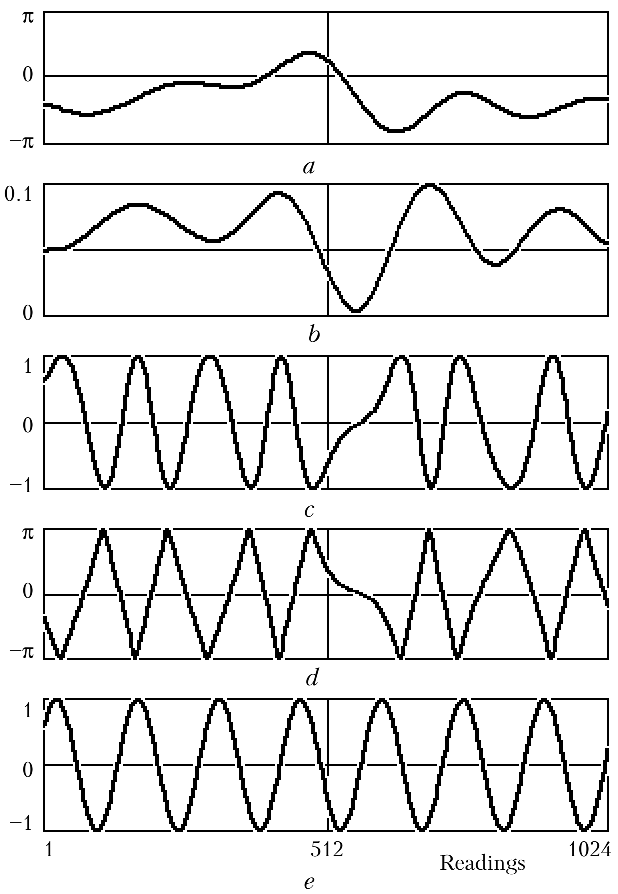

Fig. 1. Transformations of the signal with the monotonic phase: monotonic phase N(x) without linear component (a), derivative of the monotonic phase N(x) (b), signal cos N(x) (c), arccos{cos N(x)} (d), harmonic oscillation obtained from the signal cos N(x) via the compression–extension operation **Œ** (e).

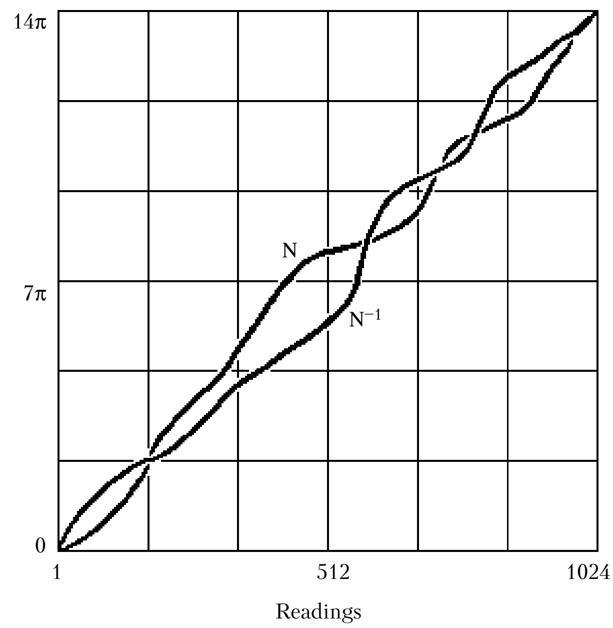

Fig. 2. Monotonic phase N and inverse phase N^{-1} for the signal shown in Fig. 1c.

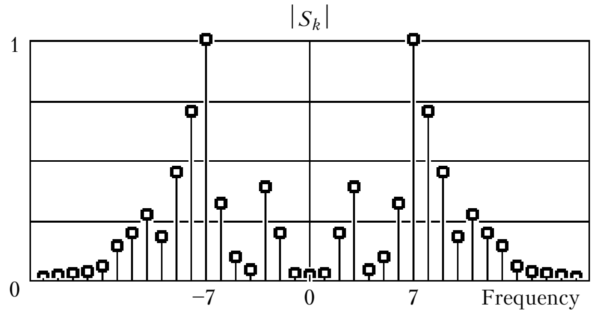

Fig. 3. Absolute value of Fourier transform of signal shown in Fig. 1e, obtained from signal cos N(x), shown in Fig. 1c, using the **Œ**-operation. Mode is at the frequencies equal to the number of periods of the signal cos N(x). The amplitude of the spectrum is low nearby the origin. The scale along the ordinate is logarithmic.

The effective signal (17) as well as the analytic signal (3) define the amplitude and phase in global sense, they depend on signal changes on all axis, from infinity to infinity.[4] The monotonicity is capable of local definition. Both approaches are give the equal results at the unit amplitude of the real signal $U(x)$. In this case the monotonic phase can be found* to arbitrary constant without integral transform **H** as follows

$$\mathrm{N}(x) = \int \left| \frac{\mathrm{d} \arccos U(x)}{\mathrm{d}x} \right| \mathrm{d}x = \int \frac{|U'(x)|}{\sqrt{1-U^2(x)}} \mathrm{d}x . \quad (20)$$

Figure 1 shows the numerical realization of this equation.

What is interesting, Eq. (20) is applicable to binary signals as well. Let, for example, the signal cosN(x) with the monotonic phase is subjected to some nonlinear transformation

$$\mathbf{B}\cos\mathrm{N}(x) = \begin{cases} 1, \cos\mathrm{N}(x) \ge 0, \\ -1, \cos\mathrm{N}(x) < 0. \end{cases} \quad (21)$$

Then, according to Eq. (20) we obtain

$$\left| \frac{\mathrm{d}}{\mathrm{d}x} [\arccos \mathbf{B} \cos\mathrm{N}(x)] \right| = \pi \sum_{n=0}^{N} \delta[\mathrm{N}(x) - \pi(2n+1)/2] \to$$

$$\xrightarrow{\int} \pi \sum_{n=0}^{N} Y[\phi(x) - \pi(2n+1)/2] = \mathrm{N}_Y(x) , \quad (22)$$

where N is the number of periods of the signal.

Thus, the reconstructed phase $\mathrm{N}_Y(x)$ is a sum of the Heaviside functions; it is the unit-step function coinciding with the initial phase N(x) at the points, where the initial phase is a multiple of the odd number of $\pi/2$, and the reconstructed phase is constant between these points:

* This result was obtained in cooperation with Yu.N. Isaev.

$$N_Y(x) = \begin{cases} N(x_n),\ n: N(x_n) = \pi(2n+1)/2 \\ \pi(2n+1)/2,\ x_n < x < x_{n+1} \end{cases} \Bigg|_{n=0,1,2,\ldots} . \quad (23)$$

Numerical realization of the operations (20)–(22) has shown the absence of significant difficulties.

3. Optimal calculation of the Hilbert transform

The algorithm for calculation of the Hilbert transform based on the direct calculation of the discrete convolution is described in Ref. 30. The direct transformation looks like:

$$V(k\tau) = \begin{cases} +\dfrac{2}{\pi}\displaystyle\sum_n \dfrac{U(n\tau)}{k-n},\ k \text{ is even, } n \text{ is odd;} \\ +\dfrac{2}{\pi}\displaystyle\sum_n \dfrac{U(n\tau)}{k-n},\ k \text{ is odd, } n \text{ is even,} \end{cases} \quad (24)$$

while the inverse transformation has the symmetric form

$$U(n\tau) = \begin{cases} -\dfrac{2}{\pi}\displaystyle\sum_k \dfrac{V(k\tau)}{n-k},\ n \text{ is even, } k \text{ is odd,} \\ -\dfrac{2}{\pi}\displaystyle\sum_k \dfrac{V(k\tau)}{n-k},\ n \text{ is odd, } k \text{ is even,} \end{cases} \quad (25)$$

where τ is the half-cycle of the highest frequency in the spectrum of the transformed function; the indices k and n vary within the interval of definition of the function.

By this algorithm, the discrete representation of the initial continuous signal is transformed into the discrete representation of its Hilbert transform, which is close to its continuous prototype provided that

– readings of the initial signal are obtained according to the Kotelnikov theorem;

– the initial signal is well defined by its readings on a finite support in the sense that beyond the support it does not differ significantly from zero by some norm.

Realization of Eqs. (24) and (25) as a Fortran program[15] was recognized as new by GosFAP in 1978. Numerical experiments have shown that the program is rather efficient when the two conditions mentioned above are fulfilled. However, in the general case one cannot expect that the accuracy of such transformation will be high, because the Hilbert transform is defined on the infinite interval and physically unrealizable.

Another possibility of numerical realization of the Hilbert transform follows from its properties in the frequency region (6) and reduces to multiplication of the signal spectrum by a signum function. One of the early papers (Ref. 25) uses an approximate harmonic analysis for realization of this approach.

The FFT is now the most efficient and well developed algorithm for Fourier transform of a series of readings.[14,25,36] It could be used as a basis for realization of the Hilbert transform and other types of linear filters. However, the FFT algorithm assumes that readings are given on a circle; hence, they can correspond to only a periodic function.

Thus, there is inconsistency between the infinite domain of definition of the functions of trigonometric basis, the Hilbert transform operator, and the finite domain of definition of the signal. This domain is two-dimensional and, possibly, multiply-connected, if we deal with demodulation of interferograms.[16,18]

The described inconsistency shows itself as a ripple of the transformed function and, of course, of its phase. The widely used method for suppression of this edge effect in the case of spectral estimates is application of smoothing windows. There are some examples of the use of windows in interferometry as well. In Ref. 31, when applying the Hanning window,[28] the ripples were decreased roughly threefold. However, in that case the object phase was constant, though the values of the signal itself did not coincide at the edges of the interval, because the number of interference fringes was not integer. In addition, there was no noise in that example. In the general case, windows are inefficient for suppression of the edge effect in the reconstructed phase. Windows significantly decrease the signal amplitude at the edges of the interval of definition, and the filtering results in relative increase of the noise level in these intervals. This fact was first noticed in Ref. 38, in which the interferogram was multiplied by the Hanning window. The edge effect in that case was marked at 20% of the length of the interval of definition, and the maximal value of phase ripples is larger than 0.2π rad.

An adequate method for solution of the discussed problem is optimal periodic continuation of the signal beyond the domain of definition with the following application of FFT.

In numerical experiments it was found[8,19] that the monotonic character of the phase or the possibility to reduce it to monotonic due to evenness and periodicity of the cosine (2) is sufficient for introduction of the associated effective signal. This opens wide possibilities for selection of rectilinear and curvilinear cross sections in the domain of definition of interferogram. If the selected scanning curve is closed and belongs to the domain of definition, then the problem of continuation does not arise, because the cross section is a periodic function of the scanning parameter. However, cross sections usually break at the external boundaries of the domain of definition. Besides, at multiple connection, the cross section breaks appear at internal boundaries. As a result, it can consist of several finite fragments.

Now we consider algorithms for transforming such fragments into a periodic function, for which the methodically exact algorithm of the Hilbert transform is possible. To derive the optimality criterion, we assume that $U(x)$ is a real bounded function defined for all $x \in (-\infty, \infty)$, and the signal consists of the known fragments of this function. Assume also that the continued function $\breve{U}(x)$ coincides with $U(x)$ within the signal. Introduce the square-integrable function $\Omega(x)$ having the low-frequency spectrum nonoverlapping with the spectra of the functions $\breve{U}(x)$ and $U(x)$. Take into account the property,[24] allowing the low-frequency oscillation to put before the operator of the Hilbert transform

$$\mathbf{H}\Omega(x)\ U(x) = \Omega(x)\ \mathbf{H}\ U(x) = \Omega(x)\ V(x), \quad (26)$$

and the fact that the energies $U(x)$ and $V(x)$ are equal. Then the following equality is valid

$$\int_{-\infty}^{\infty} \Omega^2(x)[U(x) - \breve{U}(x)]^2\,\mathrm{d}x = \int_{-\infty}^{\infty} \Omega^2(x)\{\mathbf{H}[U(x) - \breve{U}(x)]\}^2\,\mathrm{d}x. \quad (27)$$

Let the function $\Omega(x)$ tend to unity within the known fragments of the signal and to zero beyond these fragments. In this case, the left-hand side of Eq. (27) tends to zero by definition, and the spectrum of the function $\Omega(x)$ extends. When the spectrum becomes so wide that it overlaps the spectra of $\breve{U}(x)$ and $U(x)$, the equality (27) becomes invalid. For the equality to be violated at as small as possible value of its left-hand side, it is necessary to provide for as narrow-band functions $\breve{U}(x)$ and $U(x)$ as possible.

A posteriori one can only decrease the width of the spectrum of the function $\breve{U}(x)$ by the following operations of continuation:

– more smooth interpolation of individual fragments of the signal inside the domain of definition,

– more smoth extrapolation beyond the domain of definition,

– selection of the functional dependence providing for the necessary smoothness.

Thus, for the Hilbert transform of the continued function $\breve{U}(x)$ to be the closest to the true function $U(x)$, it is necessary for the continuation operation to provide for the minimal width of the spectral band of the function $\breve{U}(x)$ at the given fragments of $U(x)$. Just this is the optimality criterion.

As structure restrictions, we should take into account the properties of the basic FFT algorithm.[25,36] This operation is defined on a circle. Therefore, the cross section should be continued periodically to the whole infinite axis. Besides, we must keep in mind that the FFT algorithm deals with arrays of quite definite length. This is connected with factorization of a number equal to the array length and with calculation of the Fourier transform of every elementary array, whose length is equal to the corresponding factor. Usually this factor equals two, and the length of the whole array is some integer power of two. However, any compound number can be uniquely represented as a product of prime numbers accurate to their arrangement. The prime factor cannot be large, since this decreases the algorithm speed. Reference 36 describes an algorithm and presents the Fortran program using the first ten prime numbers (1, 2, 3, 5, 7, 11, 13, 17, 19, 23) for factorization. The set of compound numbers formed from these factors will be denoted as P_{10}.

A priori information includes a fundamental property of monotonicity. The phase $\mathrm{N}(x, y)$ in cross section has a significant linear or square (or close to them) component. Just this causes the fringe structure of an interferogram. The linear continuation of this component with conservation of continuity corresponds to the introduced optimality criterion, since the effective width of the spectrum S_k is determined by the derivative of the signal phase (9). Consequently, at continuation the conditions should be created for appearance of extra interference fringes or their parts, whose width should slightly differ from the width of given neighboring interference fringes.

The well-known theorem on convergence of Fourier series connects the rate of convergence with the number m of continuous derivatives of the function represented by the series, $S_k = o(1/k^{m+1})$ at $k \to \infty$. It is clear from this expression that jumps between the known fragments $U(x)$ and the fragments $\breve{U}(x)$ obtained as a result of continuation are undesirable. But this situation is characteristic for the iteration method of continuation.[29,35] Either a significant number of iterations is performed and fragments are joined smoothly in the case of convergence or only several iterations are performed due to deficit of time and breaks arise between the functions $\breve{U}(x)$ and $U(x)$ at the joining points.

A technique of complementing the interference fringes by a sinusoid in some cross sections of an interferogram is described in Ref. 37; the initial phase, frequency, and amplitude of the sinusoid are determined from the given fragments. In this technique, it is necessary to determine the positions of extreme points in the cross section. But under the noisy conditions, this operation is incorrect and smoothness of the extension fails to be realized.

To complement the cross section, we take its own fragments as a model and minimize some smoothness functionals on the set of cross section readings.

Consider one of the possible algorithms of realization of this method. The continuation is constructed through shift of cross section fragments beyond the domain of definition. Assume that the cross section to be continued is defined on the interval $[1, n]$. First, we continued the right-hand edge of the cross section by some number of readings r, which provides for the minimum of the functional

$$L_r = \sum_{\substack{i=n-k+1 \\ r\in(k,n-k)}}^{n} |U(i) - U(i-r)| = \min. \quad (28)$$

Then continued the left-hand edge of the cross section by l readings according to the condition

$$L_l = \sum_{\substack{i=1 \\ l\in(k,n-k)}}^{n} |U(i) - U(i+l)| = \min. \quad (29)$$

The minimum can be sought by the simple trial-and-error method. The distances between both the two functions and their differences $\Delta U(i)$ are minimized simultaneously; for example, at $k = 2$ we obtain

$$\min = |U(l) - U(l+1)| + |U(2) - U(l+2)| >$$
$$> |U(2) - U(1)| - |U(l+2) - U(l+1)| =$$
$$= |\Delta U(1) - \Delta U(l+1)|. \quad (30)$$

The new edges are joined in the similar way to obtain a periodic function, but the smoothness functional depends on two parameters

$$L_{pq} = \sum_{\substack{i=1 \\ p\in(0,n-k) \\ q\in(0,l-k)}}^{n} |U(r+i+p-1) - U(i+q)|. \quad (31)$$

A conditional minimum L_{pq} such that the whole length of the continued cross section $n_c = n + l + p + q$ belongs to the set of numbers P_{10} is also sought by the trial-and-error method. Therefore, the sufficient density of these numbers in the interval $[n, 3n]$, $r \in n_c$, is of principal significance and the efficiency of the method depends on it. The numbers for the most commonly used range from 2^7 to 2^8 are tabulated below.

128	130	132	133	135	136	138	140	143	144	147
150	152	153	154	156	160	161	162	165	168	169
170	171	175	176	180	182	184	187	189	190	192
195	196	198	200	204	207	208	209	210	216	220
224	225	228	234	240	242	243	245	250	252	256

The numbers of the set P_{10} increase exponentially. Consequently, the possibility of continuation to some interval decreases starting from some number. The possibility of continuation can be determined as a ratio of the quantity of numbers from P_{10} in a given interval to the quantity of the natural scale numbers in this interval (Fig. 4). Figure 5 exemplifies continuation of the signal cross section by the described algorithm.

From the above description of the method it can be concluded that the method corresponds to the introduced optimality criterion at fixed k and n for $n_c \in P_{10}$ and in the class of functions of the cross section to be continued. Numerical experiments have shown that the proposed method of continuation of interferogram cross sections beyond the domain of definition solves the formulated problem and diminishes markedly the edge effects in the reconstructed phase.

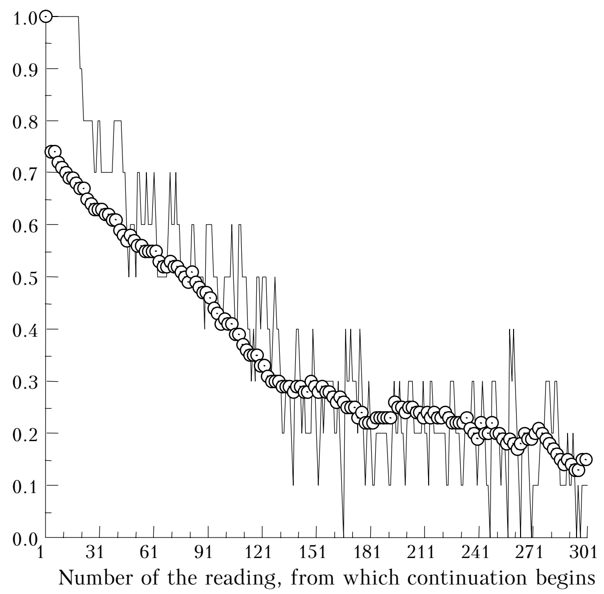

Fig. 4. Possibility of continuation of interferogram cross section by ten readings (broken line) and by hundred readings (circles) at factorization of the array dimension into prime numbers {1, 2, 3, 5, 7, 11, 13, 17, 19, 23} for FFT-algorithm.

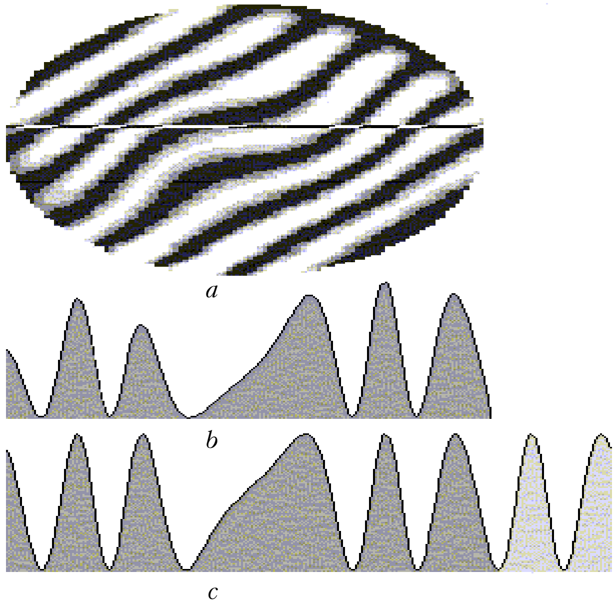

Fig. 5. Optimal continuation of the interferogram cross section: cross section is shown by inverse contrast (a); interferogram cross section (b); continued cross section, continuation is shown in light color, polynomial modulation is excluded (c).

Conclusions

Thus, we have analyzed the state of the art in definition of the amplitude and phase of the real signal and separated two important problems. They are the condition for existence of a two-band spectrum of the real signal, which also is the condition for possible introduction of the analytic signal, and the optimal calculation of the Hilbert transform, which is necessary for numerical realization. The following results have been finally obtained.

– The condition of dispersion causality using the idea of the effective width has been formulated. It has been found that the phase monotonicity provides for fulfillment of the condition of dispersion causality for the Fourier transform of the complex signal with the constant amplitude.

– A complex effective signal has been introduced, which can be applied when the spectrum is not rigorously two-band. The iteration method has been developed for its realization.

– For signals with the monotonic phase and constant amplitude a local phase definition has been proposed. The monotonic phase can be found directly from the real signal, without the complex signal, and this method is applicable to binary signals as well.

– An algorithm has been developed for calculation of the Hilbert transformation; this algorithm is based on the optimal periodic continuation of the function to be transformed according to the criterion of minimal width of the Fourier spectrum by shift of fragments of the function beyond its domain of definition.

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